An Introduction to Data Envelopment Analysis

A Tool for Performance Measurement

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I dedicate this book to my parents,
Ramakrishnan and Parvatha Meenakshi

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Foreword

Our society increasingly feels the need to implement transparent and consistent decision processes. Hence, the citizens are more and more confronted with evaluation procedures which support their public and private decisions. The choice of a secondary school for their children is supported by league tables which record and assess the performance of the schools in their neighbourhood, research and education at the universities is subject to elaborate output-based evaluation procedures, and alternative infrastructural improvements (alternative motorway and railway trajectories, alternative airport extensions) are subject to environmental assessment procedures. Public authorities in the European Union are obliged, whenever they issue a call for tenders, to publish a list of the relevant criteria ranked in a decreasing order of importance. With the scores assigned to the alternatives and with proper criterion weights, decisions can be made with due regard to the objectives of the decision makers and on the basis of state-of-the-art knowledge of technical, ecological, financial and other experts.

There is a rich variety of methods and techniques for the design of transparent decision processes. Multi-Criteria Decision Analysis (MCDA), for instance, can be used to identify a preferred alternative, to rank the alternatives in a decreasing order of preference, or to classify the alternatives into a small number of categories. Although these techniques clearly explain the rank-order positions of the alternatives, they do not always indicate how a particular alternative could reasonably improve its position. This is typically a strong feature of Data Envelopment Analysis (DEA). In principle, DEA is concerned with a number of alternative Decision-Making Units (DMUs). Each of them is analyzed separately via a mathematical-programming model which checks whether the DMU under consideration could improve its performance by decreasing its input and increasing its output. The improvement is pursued until the boundary of the convex hull of the other DMUs is reached. A DMU which cannot improve its performance is efficient or non-dominated. Otherwise, it is dominated by a convex combination of other DMUs. In summary, possible improvements for a particular DMU are indicated, not in an arbitrary direction, but on the basis of the performance of the more successful efficient DMUs.

Data Envelopment Analysis is the major subject of this book. Dr Ramanathan reviews the theoretical foundations, the algorithmic implications, and the computational implementations. Thereafter he amply discusses several applications of DEA in the public sector. Hence, this book is an excellent tool for practitioners who are interested in the merits and the pitfalls of the technique. It will prevent naïve applications without deeper reflections on what the technique actually means for those who are subject to an evaluation by DEA.

Dr Ramanathan spent two long periods in the Faculty of Information Technology and Systems of the Delft University of Technology (September 1995–February 1996 and May–September 1999) where he significantly contributed to my project 'Multi-Criteria Decision Analysis and Multi-Objective Optimization'. He devoted his time and energy to various subjects, in the first period to the Multiplicative AHP and to Fair Allocation of Resources, in the second period to Data Envelopment Analysis. Several successful publications appeared later on in the literature. He has a lot of energy indeed, and his research is an example of inventiveness, diligence, and accuracy. Moreover, it was a pleasure to discuss with him the basic assumptions and the objectives of our efforts to address administrative problems with mathematical tools. We are operating in the wider fields of Operations Research and Decision Analysis, where scientists are sometimes heavily criticized for their quantitative approach of administrative problems. Nevertheless, we believe that, both, an unbiased collection and a careful analysis of the relevant data constitute a major contribution to the formulation and the solution of such problems.

It is a real pleasure to congratulate Dr Ramanathan on his achievements. I expect that many scientists and practitioners will benefit from his thorough studies and his suggestions for further research.

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The Netherlands

With great sadness, I have to inform readers that Professor Freerk A. Lootsma passed away on Friday, 16 May 2003. He contributed significantly to the field of Multi-Criteria Decision Making and was a great motivator for researchers like me in his fields of specialization. He was a thorough gentleman, and in his untimely passing I have lost a good friend and mentor.

R. Ramanathan

Preface

Data Envelopment Analysis (DEA) is a mathematical programming technique that has found a number of practical applications for measuring the performance of similar units, such as a set of hospitals, a set of schools, a set of banks, etc. This book has been developed as an introductory textbook on DEA with some exercises at the end of many chapters. The book has been developed from the teaching and research experiences of the author while he was associated with the Indira Gandhi Institute of Development Research, Mumbai, India, the Delft University of Technology, Delft, The Netherlands, and the Helsinki University of Technology, Finland.

Though DEA is essentially a mathematics-oriented subject, attempt has been made in the book to minimize rigorous mathematical treatment and provide the concepts descriptively. While the book is meant for a general audience and beginners with minimum knowledge on mathematics, a major prerequisite for the readers of this book is knowledge on linear programming.

The book begins with the basics of efficiency measurement and frontier analysis. A linear programming formulation is employed to explain DEA in the next chapter. Various programming aspects of DEA, such as dual formulations, are also described in this chapter. Chapter 3 provides a treatment of economies of scale in DEA. Some of the extensions on DEA, namely multiplicative models, time series analysis, treatment of non-discretionary variables, categorical variables, and incorporating judgements in DEA models have been covered in Chapter 4. For those interested in knowing more about these extensions, appropriate references on research articles have been provided in this chapter. Exercises are included at the end of each chapter. Computational features of DEA, details on DEA software, and Internet support facilities have been covered in Chapter 5. Some DEA software available freely (for academic use) on the Internet have been described and illustrated. Use of spreadsheets such as MS Excel to solve DEA programs has been illustrated. A brief literature survey and references to DEA bibliographies are provided in Chapter 6. To give a practical flavour to the book, a number of real life performance measurement studies, especially from the research carried out by the author and published in reputed international journals, have been described in this chapter. The applications cover studies in the Netherlands, India and the world. The practical applications described in the book have appeared as research articles in *Energy Policy*, *International Journal of Global Energy Issues*, *Energy—The International Journal*, *OPSEARCH*, and *Indian Journal of Transport Management*. Other applications reported in the literature have been used to provide additional exercises. Some practical issues pertaining to the application of DEA and some research issues on DEA are discussed in the last chapter. A list of articles referred in the book is available as a separate reference list. For the benefit of students, some problems from the exercises have been solved, and the solutions are provided at the end of the book.

Because of the popularity of Internet and the availability of several informative Internet sites on DEA, several Internet web sites have been referred throughout this book. Most the sites have been accessed in the years 2001 and 2002 (at the latest in April 2002).

It is hoped that this book will be useful for those engaged in research and applications of operations research and performance measurement (including those who work in the field of economics). DEA is likely to be offered as a full course or as part of advanced quantitative courses in the departments of Business, Management Science, Quantitative Techniques, Industrial Engineering, Industrial Management or Mathematics of many universities and institutions. With the growing numbers of basic and advanced courses on DEA in many universities and institutions across the world, including the USA, Europe and Asia, it is expected that the book will have a worldwide audience, including students, researchers and practising consultants, engaged in operations research and performance measurement.

Acknowledgements

This book is based on my teaching experience and the materials I prepared for teaching in the Indira Gandhi Institute of Development Research (IGIDR), Mumbai, India, and the Helsinki University of Technology, Finland. I would like to acknowledge my students in both the Institutions who helped me improve the contents of this book through their helpful suggestions and feedback on my courses. I also wish to acknowledge the critical comments and suggestions provided by the reviewers of the book.

My interest in DEA started long ago when I worked with my research supervisor Professor L.S. Ganesh at the Indian Institute of Technology, Madras, India, though I could not devote much attention to this subject at that time. While I carried out some DEA studies subsequently when I was associated with IGIDR, the main thrust to my DEA studies happened when I visited the Delft University of Technology, Delft, The Netherlands. I thank the Delft University of Technology for offering me the senior research fellowship, and, the late Professor Freerk Lootsma for hosting me in his department and for arranging funds to present our research findings on the application of DEA to Dutch Schools in a conference in the International Institute of Applied Systems Analysis, Austria.

My knowledge on DEA further improved during my stay in Helsinki University of Technology, Finland. I thank the University for offering me the Visiting Professorship. I thank Professors Ahti Salo and Raimo Hamalainen of the Helsinki University of Technology, for their helpful views on DEA. I would like to thank Professor Pekka Korhonen of the Helsinki School of Economics and Business Administration for his insight, helpful suggestions and comments on DEA during our interactions.

I thank my present institution, the Sultan Qaboos University, for the assistance during the final stages of the preparation of this book. I thank the members of Sage Publications for the excellent typesetting and organization of this book.

My wife Usha and children Bharadwaj and Nila played an absolutely vital role towards the completion of this book. Many a times they have tolerated my absent-mindedness and created a warm family atmosphere that drives away the loneliness of scientific research.

List of Abbreviations

The following abbreviations pertain to the theory of DEA and are used in many parts of the book.

DEA	Data Envelopment Analysis
DMU	Decision-Making Unit
DM	Decision-Maker
LP	Linear Programming
CCR	Charnes, Cooper and Rhodes
BCC	Banker, Charnes and Cooper
LHS	Left-Hand Side
RHS	Right-Hand Side
CAP	Capital Employed
VA	Value Added
EMP	No. of Employees
MPI	Malmquist Productivity Index
FDH	Free Disposal Hull
MCDM	Multi-Criteria Decision-Making
DRS	Decreasing Returns to Scale
IRS	Increasing Returns to Scale
VRS	Variable Returns to Scale
CRS	Constant Returns to Scale
NDRS	Non-Decreasing Returns to Scale
NIRS	Non-Increasing Returns to Scale
MPSS	Most Productive Scale Size
DEAP	A Data Envelopment Analysis (Computer) Program
EMS	Efficiency Measurement System
GAMS	General Algebraic Modelling System

The following abbreviations are used only in one chapter/section of a chapter.

STU	State Transport Undertaking
PKM	Passenger Kilometres
TKM	Tonne-Kilometres
TJ	Tera Joules
IS	Information System
OECD	Organization for Economic Cooperation and Development
MW	MegaWatts
kcal	kilocalories
GWh	GigaWatt-hour
VBA	Visual Basic for Applications
GDP	Gross Domestic Product
CRA	Comparative Risk Assessment
LLE	Loss of Life Expectancy
GLE	Gain of Life Expectancy
PV	Photovoltaic
CO_2	Carbon dioxide
FOSS	Fossil Fuel Energy Consumption
NFOSS	Non-Fossil Fuel Energy Consumption

1

Introduction

Data Envelopment Analysis (DEA) is a methodology based upon an interesting application of linear programming. It was originally developed for performance measurement. It has been successfully employed for assessing the relative performance of a set of firms that use a variety of identical inputs to produce a variety of identical outputs. The principles of DEA date back to Farrel (1957). The recent series of discussions on this topic started with the article by Charnes et al. (1978). A good introduction to DEA is available in Norman and Stoker (1991). Cooper et al. (2000) provide recent and comprehensive material on DEA.

1.1 Decision-making Units

Data Envelopment Analysis is a linear programming-based technique for measuring the performance efficiency of organizational units which are termed Decision-Making Units (DMUs). This technique aims to measure how efficiently a DMU uses the resources available to generate a set of outputs (Charnes et al. 1978). Decision-making units can include manufacturing units, departments of big organizations such as universities, schools, bank branches, hospitals, power plants, police stations, tax offices, prisons, defence bases, a set of firms or even practising individuals such as medical practitioners. As we shall see later in this book, DEA has been successfully applied to measure the performance efficiency of all these kinds of DMUs.

Most of these DMUs are non-profit organizations, where the measurement of performance efficiency is difficult.[1] Note that the efficiency of commercial organizations can be assessed easily by their yearly profits, or their stock market indices. However, such measurable factors are not applicable to non-profit organizations. The problem is complicated by the fact that the DMUs consume a variety of identical inputs and produce a variety of identical outputs. For example, schools can have a variety of inputs, which are the same for each school—quality of students, teachers, grants, etc. They have a variety of identical outputs—number of students passing the final year, average grade obtained by the students in their final year, etc.

The performance of DMUs is assessed in DEA using the concept of efficiency or productivity, which is the ratio of total outputs to total inputs. Efficiencies estimated using DEA are *relative*, that is, relative to the best performing DMU (or DMUs if there is more than one best-performing DMUs). The best-performing DMU is assigned an efficiency score of unity or 100 per cent, and the performance of other DMUs vary, between 0 and 100 per cent relative to this best performance.

1.2 Basic Concepts of Efficiency Measurement

As mentioned earlier, the basic efficiency measure used in DEA is the ratio of total outputs to total inputs.

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} \tag{1.1}$$

[1] Over the past years, DEA has been applied to profit-making organizations also. This is partly because profit *per se* is not a good indication of the potential for improvement within an organization, and because other factors are necessary for a holistic assessment of performance.

1.2.1 Case of Single Input and Single Output

Let us consider the performance of four firms on the basis of one input measure (capital) and one output measure (value added)[2] as listed in Table 1.1.

Table 1.1 Performance of four firms

Firm	*Input* (*Fixed* + *Working Capital*) (*$million*)	*Output* (*Value Added*) (*$million*)
A	8.6	1.8
B	2.2	0.2
C	15.6	2.8
D	31.6	4.1

Note that the firms consume differing quantities of inputs and produce differing levels of outputs. Given the data, how can we compare their performance?

The easiest approach is to use ratios. As there is only one input and only one output, ratio computation is simple. Details are shown in Table 1.2.

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} = \frac{\text{Value Added}}{\text{Capital Employed}}$$

Table 1.2 Comparison of performance of the firms

Firm	*Capital Employed* (*$million*)	*Value Added* (*$million*)	*Value Added per Capital Employed*
A	8.6	1.8	0.209
B	2.2	0.2	0.091
C	15.6	2.8	0.179
D	31.6	4.1	0.130

Firm A has the highest value added per unit of capital employed, while Firm B has the lowest. As Firm A has the highest ratio, we can compare the performance of other firms relative to that of

[2] The term ‘value added’ is an economic term, and is the money value of all intermediate inputs in a firm subtracted from the output.

Firm A. Setting the performance efficiency of Firm A as 100 per cent, we can calculate the *relative efficiencies* of the other firms, as shown in Table 1.3.

Table 1.3 Relative efficiencies of firms

Firm	*Value Added per Capital Employed*	*Relative Efficiency (%)*
A	0.209	100.0
B	0.091	43.4
C	0.179	85.8
D	0.130	62.0

A fundamental assumption behind the computation of relative efficiency is that if a given firm, A, is capable of producing $Y(A)$ units of output using $X(A)$ of inputs, then other firms should also be able to do the same if they were to operate efficiently.

We can set *Performance Targets* for inefficient firms to enable them to reach 100 per cent relative efficiency in comparison with Firm A, the most efficient. Firm A has operated in an environment similar to the others and hence using its performance as a benchmark is realistic. *Input Target* for Firm B is the amount of capital employed that will enable the firm to have the same ratio of value added to capital employed as Firm A.

$$\text{Input Target} = \text{Actual Input} \times \text{Relative Efficiency}/100 \quad (1.2)$$

For Firm B,

$$\text{Input Target} = 2.2 \times 0.434 = 0.955$$

This means that if Firm B operates using \$0.955 million as input, and produces \$0.2 million as value added output, then it will be considered as efficient as Firm A.

For inefficient firms, input target will be less than *Actual Input*. The difference between actual input and input target is *Input Slack*.

For Firm B,

$$\begin{aligned} \text{Input Slack} &= \text{Actual Input} - \text{Input Target} \quad (1.3) \\ &= 2.2 - 0.955 \\ &= 1.245 \end{aligned}$$

Input Slack can also be expressed as a percentage.

$$\text{Input Slack Percentage} = \frac{\text{Input Slack}}{\text{Actual Input}} \times 100 \qquad (1.4)$$

For Firm B,

$$\text{Input Slack Percentage} = \frac{1.245}{2.2} \times 100 = 56.6$$

Thus, if Firm B has to be as efficient as Firm A, it should produce the same output using 57 per cent less inputs.

Input targets and slacks for other firms can be computed similarly.

Using a similar logic, we can compute *Output Targets* and *Output Slacks*.

$$\text{Output Target} = \frac{\text{Actual Output}}{\text{Relative Efficiency}/100} \qquad (1.5)$$

$$\text{Output Slack} = \text{Output Target} - \text{Actual Output} \qquad (1.6)$$

$$\text{Output Slack Percentage} = \frac{\text{Output Slack}}{\text{Actual Output}} \times 100 \qquad (1.7)$$

For Firm B,

$$\text{Output Target} = \frac{\text{Actual Output}}{\text{Relative Efficiency}/100} = \frac{0.2}{0.434} = \$0.46 \text{ billion}$$

$$\begin{aligned} \text{Output Slack} &= \text{Output Target} - \text{Actual Output} \\ &= 0.46 - 0.2 \\ &= \$0.26 \text{ billion} \end{aligned}$$

$$\begin{aligned} \text{Output Slack Percentage} &= \frac{\text{Output Slack}}{\text{Actual Output}} \times 100 \\ &= \frac{0.26}{0.2} \times 100 \\ &= 130 \end{aligned}$$

Thus, if firm B is to achieve the same efficiency as Firm A, it should increase its output by $0.26 million or by 130 per cent for the same level of input (capital).

1.2.2 Case of Single Output and Two Inputs

In practice, no firm consumes only one input to produce a single output. Let us now add one more input—number of employees, as listed in Table 1.4.

Table 1.4 An additional input for four firms

Firm	*No. of Employees ('000)*
A	1.8
B	1.7
C	2.6
D	12.3

How do we compare the performance of the firms now? We can still use ratios, but we now have two ratios. Calculations are shown in Table 1.5. Throughout this textbook, whenever this example is considered, value added and capital employed are always measured in million dollars, and employees in thousands.

Table 1.5 Comparison of performance of firms with one output and two inputs

Firm	*Value Added per Capital Employed*	*Value Added per Employees*
A	0.209	1.000
B	0.091	0.118
C	0.179	1.077
D	0.130	0.333

Note that Firm A has produced the largest value added per unit of capital employed, while Firm C has produced the maximum value added per employee. As we do not know which ratio is more crucial, we cannot say Firm A is more efficient than Firm C or otherwise. What we can conclude is that Firms B and D are not as efficient as Firms A and C, because their ratios of output to input (i.e., value added to capital or value added to employees) are lower.

1.3 Graphical Description—Frontier Analysis

One way to tackle the problem of interpreting different ratios, for problems involving two inputs and one output, is through graphical analysis. Let us plot the two ratios for each firm as shown in Figure 1.1.

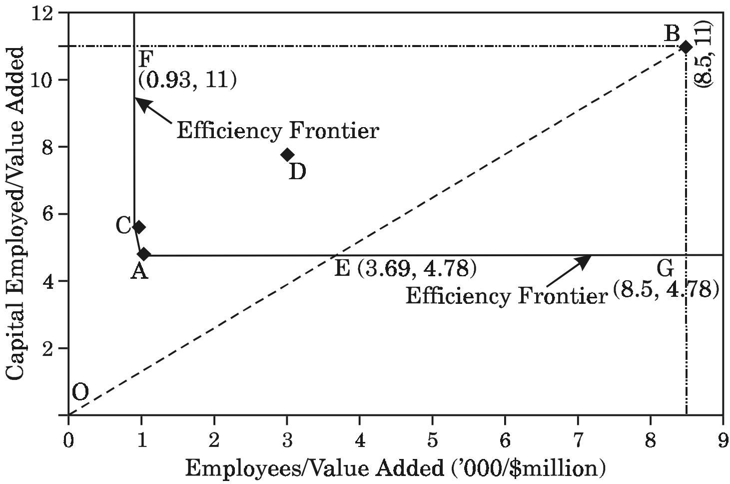


Figure 1.1 Frontier analysis in DEA

As we have two inputs and one output, we have used ratios of input to output as axes in Figure 1.1. Therefore, the values used for the two axes are the reciprocals of the values reported in Table 1.5. The reciprocals are shown in Table 1.6. The firms that are more efficient consume lower levels of input per unit of output and hence lie closer to the origin. As discussed previously, Firms A and C are more efficient than the others. Given these ratios, one can draw an *Efficiency Frontier*, as the line joining the more efficient firms and the vertical and horizontal lines connecting them to the two axes. The efficiency frontier is indicated in Figure 1.1. It represents a standard of performance that the firms not on the frontier should try to achieve. Firms on the frontier (Firms A and C here) are considered 100 per cent efficient.

Table 1.6 Performance of firms with one output and two inputs (input/output ratios)

Firm	*Capital Employed/ Value Added*	*Employees/ Value Added*
A	4.778	1.000
B	11.000	8.500
C	5.571	0.929
D	7.707	3.000

Such an analysis using the efficiency frontier is often termed *Frontier Analysis* (Farrel 1957).[3] This efficiency frontier forms the basis of efficiency measurement. The efficiency frontier envelops the available data. Hence, the term *Data Envelopment Analysis*. In the DEA literature, Firms A and C are called efficient firms while Firms B and D that do not lie on the efficiency frontier are called inefficient firms.

Firms A and C lie on the efficiency frontier, and hence are the most efficient. Note that this does not mean that their performance cannot be improved. It may or may not be possible. The available data does not give any idea regarding the extent to which their performance can be improved. These are the best firms with regard to the data we have. As no other firm shows better performance, we should assume that their performance is the *best achievable*. We rate the performance of all other firms *in relation to* this best achieved performance. Thus, we consider *relative* efficiencies, not absolute efficiencies.

1.3.1 Estimating Efficiencies of Inefficient Firms

Consider Firm B. The firm is inefficient because it does not lie on the frontier. Can we make a quantitative estimate of its efficiency, *in relation to* the performance of the best firm lying on the frontier? Firm B uses $2.2 million of capital and 1,700 employees.

[3] The economic treatment of *Frontier Analysis* is normally explained using the concept of *production possibility frontier* (Samuelson and Nordhaus 1989; Thanassoulis 1999). A *production possibility set* for the example in this section can be constructed by considering all possible combinations of the two inputs (capital employed and employees) that will result in the same, given level of output (value added), and the production possibility frontier is the frontier enveloping the production possibility set.

Let the firm's output be increased keeping the ratio of inputs (i.e., capital to employees) unchanged. In Figure 1.1, the firm will move along the line joining the origin with Firm B (dotted line OB in the figure). Obviously, the best performance possible for Firm B (retaining the same ratio of inputs) occurs at the point E, the intersection of the line OB and the efficiency frontier. The co-ordinates of the point E can be determined using the principles of analytical geometry.

Note that the best possible performance cannot go below the line AG as shown in Figure 1.1 because the line defines the best achieved performance. This is exemplified by Firm A in this case. Thus, Firm A defines the best achievable performance of Firm B. Hence Firm A is called the *peer* for Firm B. Peers are always efficient firms. Inefficient firms can try to emulate their peers in order to improve their efficiency.

Given this best possible performance, one can measure the efficiency of Firm B as the ratio of distance OE to the distance OB.

$$\text{Relative Efficiency of Firm B}$$

$$= \frac{\text{Best Possible Performance}}{\text{Actual Performance}} \tag{1.8}$$

$$= \frac{OE}{OB}$$

$$= \frac{\sqrt{3.69^2 + 4.78^2}}{\sqrt{8.5^2 + 11^2}}$$

$$= 0.4344$$

Hence, the *relative efficiency* of Firm B is 43.44 per cent. Similarly, the relative efficiency of Firm D can be estimated.

1.3.2 Performance Targets and Slacks for Inefficient Firms

Similar to the earlier exercise, we can estimate input and output targets. Let us again consider Firm B. It can move up to the efficiency frontier in at least three ways.

(*a*) If Firm B reduces the number of employees, with capital and value added being unchanged it moves to point F, where the ratio of employees to value added is 0.9286. As value added for Firm B is $0.2 million,

Input Target for Employees = 0.9286 ´ 0.2 = 0.1857

Input Slack for Employees = Actual Input – Input Target

= 1.7 – 0.1857

= 1.5143

$$\text{Input Slack Percentage for Employees} = \frac{\text{Input Slack}}{\text{Actual Input}} \times 100$$

$$= \frac{1.5143}{1.7} \times 100$$

= 89.1

(*b*) If Firm B reduces only its capital employed, keeping employees and value added unchanged, to move to point G. At this point, the ratio of capital employed to value added is 4.78. As value added for Firm B is $0.2 million,[4]

Input Target for Capital Employed = 4.78 ´ 0.2 = 0.956

Input Slack for Capital Employed = Actual Input – Input Target

= 2.000 – 0.956

= 1.244

$$\text{Input Slack Percentage for Capital Employed} = \frac{\text{Input Slack}}{\text{Actual Input}} \times 100$$

$$= \frac{1.244}{2.200} \times 100$$

= 56.5

(*c*) If Firm B reduces both capital and employees in the same ratio, or increases value added (keeping the same ratio of inputs), it moves to point E. Targets can be calculated accordingly.

[4] These calculations are the same as those carried out in the previous section, except for round-off errors.

$$\text{Output Target for Value Added} = \frac{2.2}{4.78} \equiv \frac{1.7}{3.69} = \$0.46 \text{ billion}$$

$$\begin{aligned}\text{Output Slack} &= \text{Output Target} - \text{Actual Output} \\ &= 0.46 - 0.2 \\ &= \$0.26 \text{ billion}\end{aligned}$$

$$\begin{aligned}\text{Output Slack Percentage} &= \frac{\text{Output Slack}}{\text{Actual Output}} \times 100 \\ &= \frac{0.26}{0.2} \times 100 \\ &= 130\end{aligned}$$

1.4 Strongly and Weakly Efficient DMUs

In the previous section, it was claimed that the DMUs lying on the efficiency frontier are efficient. However, further distinction among these efficient units is possible. Consider the firm corresponding to the point E in Figure 1.1. This firm is considered efficient because it lies in the efficiency frontier, but is *weakly* efficient as it has a positive slack in one of its inputs (thousand employees). Firm A is *strongly* efficient as it has no slack. Firm C is also strongly efficient.

1.5 Exercises

1. Consider only one input (capital employed) and only one output (value added). Estimate the relative efficiencies, input targets, input slacks, output targets, and output slacks for four firms A, B, C and D. Use the data given in section 1.2.
2. Consider two inputs (capital employed and number of employees) and one output (value added). Estimate the relative efficiencies, input targets, input slacks, output targets, and output slacks for the four firms A, B, C and D. Use the data given in section 1.2.
3. The table below gives the input parameters for 19 schools. A, B, C, etc., designate the schools. Plot them on a graph and

identify the most efficient schools. Draw the efficiency frontier. Which is the most inefficient school and why? Also, use your own knowledge in analytical geometry or any other idea to compute the efficiency of the following schools: B, H, Q, R and S.

School	*No. of Non-teaching Staff/Proportion of Students Passing in the Final School Exam.*	*No. of Teaching Staff/Proportion of Students Passing in the Final School Exam.*
A	9.4430	252.8089
B	96.4586	202.9064
C	152.2905	175.5855
D	71.8869	150.1708
F	87.2708	144.5602
G	68.9087	139.9097
H	35.5693	121.2792
I	92.7638	100.1946
J	166.9809	94.3587
K	18.0913	93.1002
L	99.0674	91.4172
M	38.3539	90.2501
N	11.4159	83.5886
O	60.2542	79.9985
P	122.7965	74.5208
Q	30.2807	57.2547
R	82.8648	55.9139
S	27.8246	54.5739
T	50.3536	50.1225

4. Some output parameters for hospitals designated as A, B, C, etc., are tabulated below. Plot them on a graph and identify the most efficient hospitals. Draw the efficiency frontier. What is the most inefficient hospital and why? Also, use your own knowledge in analytical geometry or any other idea to compute the efficiency of the following hospitals: B, C, H and R.

Hospital	*Average Number of Medical Surgical Intensive Care Discharges/Average Staffed Beds*	*Average Number of Medical Surgical Acute Discharges/Average Staffed Beds*
A	0.94430	25.28089
B	9.64587	20.29064
C	15.22905	17.55855
D	7.18869	15.01708
F	8.72708	14.45602
G	6.89087	13.99097
H	3.55693	12.12792
I	9.27638	10.01946
J	16.69809	9.43587
K	1.80913	9.31002
L	9.90674	9.14172
M	3.83539	9.02501
N	1.14159	8.35886
O	6.02542	7.99985
P	12.27965	7.45208
Q	3.02807	5.72547
R	8.28648	5.59139
S	2.78246	5.45739
T	5.03536	5.01225

2

Mathematical Programming Aspects of DEA

As discussed in the first chapter, performance evaluation for the case of two inputs and one output was more complicated than in the case of single input–output. Graphical analysis was used for analysing this case. However, graphical models cannot be used if we consider a greater number of inputs and outputs. Hence, a general mathematical formulation is needed to handle the case of multiple inputs and multiple outputs.

Note that the techniques of frontier analysis has been described by Farrel in 1957, but a mathematical framework to handle frontier analysis could be established only after 20 years. This mathematical formulation was provided by Charnes et al. (1978). This seminal paper provided the fundamentals of the mathematical aspects of frontier analysis. The authors also coined the term *Data Envelopment Analysis*.

2.1 Mathematical Formulation

Let us use x and y to represent inputs and outputs, respectively. Let the subscripts i and j to represent particular inputs and outputs respectively. Thus x_i represents the ith input, and y_j represent the jth output of a decision-making unit. Let the total number of inputs and outputs be represented by I and J respectively, where $I, J > 0$.

In DEA, multiple inputs and outputs are linearly aggregated using weights. Thus, the *virtual input* of a firm is obtained as the linear weighted sum of all its inputs.

$$\text{Virtual Input} = \sum_{i=1}^{I} u_i x_i, \tag{2.1}$$

where u_i is the weight assigned to input x_i during the aggregation, and $u_i \;^{3}\; 0$.

Similarly, the *virtual output* of a firm is obtained as the linear weighted sum of all its outputs.

$$\text{Virtual Output} = \sum_{j=1}^{J} v_j y_j, \tag{2.2}$$

where v_j is the weight assigned to output y_j during the aggregation. Also $v_j \;^{3}\; 0$.

Given these virtual inputs and outputs, the *Efficiency* of the DMU in converting the inputs to outputs can be defined as the ratio of outputs to inputs.

$$\text{Efficiency} = \frac{\text{Virtual Output}}{\text{Virtual Input}} = \frac{\sum_{j=1}^{J} v_j y_j}{\sum_{i=1}^{I} u_i x_i} \tag{2.3}$$

Obviously, the most important issue at this stage is the assessment of weights. This is a tricky issue as there is no unique set of weights.

For example, a school that has a good reputation of teaching humanities will like to attach higher weights to its humanities' output. A school that has a higher percentage of socially weaker groups in its students would like to emphasize this fact, assigning a greater weight to this input category. Thus, the weights assigned should be flexible and reflect the requirement (performance) of the individual DMUs.

This issue of assigning weights is tackled in DEA by assigning a unique set of weights for each DMU. The weights for a DMU are determined, using mathematical programming, as those weights which will maximize its efficiency subject to the condition that the efficiencies of other DMUs (calculated using the same set of weights) is restricted to values between 0 and 1. The DMU for which the efficiency is maximized is normally termed as the *reference* or *base* DMU or the DMU under the assessment.

2.1.1 Fractional DEA Programs

Let there be N DMUs whose efficiencies have to be compared. Let us take one of the DMUs, say the mth DMU, and maximize its efficiency according to the formula given above. Here the mth DMU is the reference DMU.

The mathematical program now is,

$$\max E_m = \frac{\sum_{j=1}^{J} v_{jm} y_{jm}}{\sum_{i=1}^{I} u_{im} x_{im}}$$

subject to

$$0 \le \frac{\sum_{j=1}^{J} v_{jm} y_{jn}}{\sum_{i=1}^{I} u_{im} x_{in}} \le 1; \quad n = 1, 2, \ldots, N$$

$$v_{jm}, u_{im} \ge 0; \quad i = 1, 2, \ldots, I; \quad j = 1, 2, \ldots, J \tag{2.4}$$

where
E_m is the efficiency of the mth DMU,
y_{jm} is jth output of the mth DMU,
v_{jm} is the weight of that output,
x_{im} is ith input of the mth DMU,
u_{im} is the weight of that input, and
y_{jn} and x_{in} are jth output and ith input, respectively, of the nth DMU, n = 1, 2, ..., N.
Note that here n includes m.

Consider the Firms A, B, C and D discussed in the first chapter. Let $v_{VA,A}$ be the weight associated with the only output (value added, *VA*) when Firm A is the reference DMU. The first subscript denotes the output, while the second subscript denotes the reference DMU. Using a similar notation, let $u_{CAP,A}$ and $u_{EMP,A}$ represent the weights of the two inputs, capital employed (CAP) and the number of employees (EMP), respectively. Thus, in DEA, the efficiency of Firm A, denoted as E_A, is defined as follows.

$$E_A = \frac{1.8v_{VA,A}}{8.6u_{CAP,A} + 1.8u_{EMP,A}} \tag{2.5}$$

This efficiency is maximized subject to the following conditions.

$$\max E_A = \frac{1.8v_{VA,A}}{8.6u_{CAP,A} + 1.8u_{EMP,A}}$$

subject to

$$0 \le E_A = \frac{1.8v_{VA,A}}{8.6u_{CAP,A} + 1.8u_{EMP,A}} \le 1$$

$$0 \le E_B = \frac{0.2v_{VA,A}}{2.2u_{CAP,A} + 1.7u_{EMP,A}} \le 1$$

$$0 \le E_C = \frac{2.8v_{VA,A}}{15.6u_{CAP,A} + 2.6u_{EMP,A}} \le 1$$

$$0 \le E_D = \frac{4.1v_{VA,A}}{31.6u_{CAP,A} + 12.3u_{EMP,A}} \le 1$$

$$v_{VA,A}, u_{CAP,A}, u_{EMP,A} \ge 0 \tag{2.6}$$

The above mathematical program, when solved, will give the values of weights u and v that will maximize the efficiency of Firm A. If the efficiency is unity, then the firm is said to be efficient, and will lie on the frontier. Otherwise, the firm is said to be *relatively* inefficient.

Note that the above mathematical program gives the efficiency of only one firm (the reference firm—Firm A here). To get the efficiency scores of the Firms B, C and D, more such mathematical programs have to be solved, considering each of them as the reference firm.

For example, to obtain the efficiency of Firm B, the following mathematical program is used.

$$\max E_B = \frac{0.2v_{VA,B}}{2.2u_{CAP,B} + 1.7u_{EMP,B}}$$

subject to

$$0 \le E_A = \frac{1.8v_{VA,B}}{8.6u_{CAP,B} + 1.8u_{EMP,B}} \le 1$$

$$0 \le E_B = \frac{0.2v_{VA,B}}{2.2u_{CAP,B} + 1.7u_{EMP,B}} \le 1$$

$$0 \le E_C = \frac{2.8v_{VA,B}}{15.6u_{CAP,B} + 2.6u_{EMP,B}} \le 1$$

$$0 \le E_D = \frac{4.1v_{VA,B}}{31.6u_{CAP,B} + 12.3u_{EMP,B}} \le 1$$

$$v_{VA,B}, u_{CAP,B}, u_{EMP,B} \ge 0 \tag{2.7}$$

2.1.2 Output Maximization and Input Minimization DEA Programs

Note that these mathematical programs are fractional programs. It is generally difficult to solve fractional programs. If they are converted to simpler formulations, such as the linear programming (LP) formats, then they can be solved easily. The simpliest way to convert these fractional programs to linear programs is to normalize either the numerator or the denominator of the fractional programming objective function!

Let us first normalize the denominator of the objective function of the fractional program that estimates the efficiency of Firm A. We obtain the following linear program (LP) for maximizing the efficiency of Firm A.

$$
\begin{aligned}
&\max 1.8v_{VA,A} \\
&\text{subject to} \\
&8.6u_{CAP,A} + 1.8u_{EMP,A} = 1 \\
&1.8v_{VA,A} - \left(8.6u_{CAP,A} + 1.8u_{EMP,A}\right) \le 0 \\
&0.2v_{VA,A} - \left(2.2u_{CAP,A} + 1.7u_{EMP,A}\right) \le 0 \\
&2.8v_{VA,A} - \left(15.6u_{CAP,A} + 2.6u_{EMP,A}\right) \le 0 \\
&4.1v_{VA,A} - \left(31.6u_{CAP,A} + 12.3u_{EMP,A}\right) \le 0 \\
&v_{VA,A}, u_{CAP,A}, u_{EMP,A} \ge 0
\end{aligned} \tag{2.8}
$$

The weighted sum of inputs is constrained to be unity in this linear program. As the objective function is the weighted sum of outputs that has to be maximized, this formulation is referred to as the *Output Maximization* DEA program.

An analogous LP formulation is possible by minimizing the weighted sum of inputs, setting the weighted sum of outputs equal to unity. That is the *Input Minimization* DEA program.

The following is the *Input Minimization* DEA program for Firm A.

$$
\begin{aligned}
&\min 8.6u'_{CAP,A} + 1.8u'_{EMP,A} \\
&\text{subject to} \\
&1.8v'_{VA,A} = 1 \\
&1.8v'_{VA,A} - \left(8.6u'_{CAP,A} + 1.8u'_{EMP,A}\right) \le 0 \\
&0.2v'_{VA,A} - \left(2.2u'_{CAP,A} + 1.7u'_{EMP,A}\right) \le 0 \\
&2.8v'_{VA,A} - \left(15.6u'_{CAP,A} + 2.6u'_{EMP,A}\right) \le 0 \\
&4.1v'_{VA,A} - \left(31.6u'_{CAP,A} + 12.3u'_{EMP,A}\right) \le 0 \\
&v'_{VA,A}, u'_{CAP,A}, u'_{EMP,A} \ge 0
\end{aligned} \tag{2.9}
$$

Because of the nature of the formulations, the optimal objective function value of the input minimization DEA program for Firm A will be the reciprocal of the optimal objective function value (i.e., the value of efficiency) of the output maximization DEA program for Firm A.

These were the original models introduced by Charnes et al. in 1978. Immediately after, the authors made a minor modification (Charnes et al. 1979). In a conventional LP, the decision variables are non-negative—they can be either zero or positive. However, the authors chose to define the decision variables of the DEA programs (i.e., the weights) to be strictly positive. They replaced the non-negativity constraints,

$$v_{VA,A}, u_{CAP,A}, u_{EMP,A} \geq 0$$

by the strict positivity constraints

$$v_{VA,A}, u_{CAP,A}, u_{EMP,A} > 0.$$

This modification restricted the input and output weights such that

$$v_{VA,A}, u_{CAP,A}, u_{EMP,A} > \varepsilon,$$

where *e* is an infinitesimal or non-Archimedean constant, usually of the order of 10^{-5} or 10^{-6}.

It must be emphasized that this non-Archimedean infinitesimal is *not* a number, and hence in principle cannot be approximated by any finite valued number. However, standard LP packages require that this infinitesimal be represented in the form of a small number.

The non-Archimedean infinitesimals (i.e., *e*s) were introduced because, under certain circumstances, the earlier model implied unit efficiency ratings for DMUs with non-zero slack variables such that further improvements in performance remained feasible. We shall discuss the use of these non-Archimedean infinitesimals in greater detail later in this chapter.

The models developed so far are called the CCR (Charnes, Cooper and Rhodes) models in the DEA literature.

2.1.3 General Form of CCR DEA Models[1]

A general output maximization CCR DEA model can be represented as follows.

$$
\begin{aligned}
&\max z = \sum_{j=1}^{J} v_{jm} y_{jm} \\
&\text{subject to} \\
&\sum_{i=1}^{I} u_{im} x_{im} = 1 \\
&\sum_{j=1}^{J} v_{jm} y_{jn} - \sum_{i=1}^{I} u_{im} x_{in} \le 0; \quad n = 1, 2, \ldots, N \\
&v_{jm}, u_{im} \ge \varepsilon; \quad i = 1, 2, \ldots, I; \quad j = 1, 2, \ldots, J
\end{aligned}
\tag{2.10}
$$

This program can be represented in matrix form as shown below.

$$
\begin{aligned}
&\max z = V_m^T Y_m \\
&\text{subject to} \\
&U_m^T X_m = 1 \\
&V_m^T Y - U_m^T X \le 0 \\
&V_m^T, U_m^T > \varepsilon
\end{aligned}
\tag{2.11}
$$

where X is the matrix of inputs and Y is the matrix of outputs.

Similarly, a general input minimization CCR DEA model can be represented as follows.

$$
\begin{aligned}
&\min z' = \sum_{i=1}^{I} u'_{im} x_{im} \\
&\text{subject to} \\
&\sum_{j=1}^{J} v'_{jm} y_{jm} = 1 \\
&\sum_{j=1}^{J} v'_{jm} y_{jn} - \sum_{i=1}^{I} u'_{im} x_{in} \le 0; \quad n = 1, 2, \ldots, N \\
&v'_{jm}, u'_{im} \ge \varepsilon; \quad i = 1, 2, \ldots, I; \quad j = 1, 2, \ldots, J
\end{aligned}
\tag{2.12}
$$

[1] This section requires knowledge of matrix algebra, and can be skipped without any loss of continuity.

The program can be represented in matrix form as shown below.

$$\begin{aligned}
&\min z' = U_m'^T X_m \\
&\text{subject to} \\
&V_m'^T Y_m = 1 \\
&V_m'^T Y - U_m'^T X \leq 0 \\
&V_m'^T, U_m'^T > \varepsilon
\end{aligned} \tag{2.13}$$

2.2 Exercises

For the exercises below, use $e = 10^{-6}$ whenever needed.

1. Write fractional programs for estimating the efficiencies of all the four firms discussed in Chapter 1.
2. Write the output maximization and input minimization CCR DEA programs for all the four firms. Consider capital employed as the only input variable and value added as the output variable.
3. Solve the CCR DEA programs listed in this section using a suitable LP software package. List the efficiencies and the weights for all the models.
4. Repeat exercises 2 and 3 considering capital employed and number of employees as the two input variables, and value added as the output variable.
5. Compute the efficiencies of the schools in the table in page 47 using an LP package. Use both output maximization and input minimization CCR DEA programs. Make use of the same data as in the previous exercise. Since there is no separate output variable, assume a dummy output of 1 for all schools.
6. Compute the efficiencies of the following hospitals using an LP package. Use both output maximization and input minimization CCR DEA programs. The data to be used is the same as in the previous exercise. As there are no separate input variables, assume a dummy input of 1 for all the hospitals.[2]

[2] The implication of inclusion of dummy inputs/outputs in a DEA program is discussed in Section 6.2.2.

Compare the results of the LPs to the results of graphical calculations in Exercise 1.5.

School	*No. of Non-teaching Staff/Proportion of Students Passing in the Final School Exam.*	*No. of Teaching Staff/Proportion of Students Passing in the Final School Exam.*
A	9.4430	252.8089
B	96.4586	202.9064
C	152.2905	175.5855
D	71.8869	150.1708
F	87.2708	144.5602
G	68.9087	139.9097
H	35.5693	121.2792
I	92.7638	100.1946
J	166.9809	94.3587
K	18.0913	93.1002
L	99.0674	91.4172
M	38.3539	90.2501
N	11.4159	83.5886
O	60.2542	79.9985
P	122.7965	74.5208
Q	30.2807	57.2547
R	82.8648	55.9139
S	27.8246	54.5739
T	50.3536	50.1225

Hospital	*Average Number of Medical Surgical Intensive Care Discharges/Average Staffed Beds*	*Average Number of Medical Surgical Acute Discharges/Average Staffed Beds*
A	0.94430	25.28089
B	9.64587	20.29064
C	15.22905	17.55855
D	7.18869	15.01708
F	8.72708	14.45602
G	6.89087	13.99097

(*Table contd.*)

(*Table contd.*)

Hospital	*Average Number of Medical Surgical Intensive Care Discharges/Average Staffed Beds*	*Average Number of Medical Surgical Acute Discharges/Average Staffed Beds*
H	3.55693	12.12792
I	9.27638	10.01946
J	16.69809	9.43587
K	1.80913	9.31002
L	9.90674	9.14172
M	3.83539	9.02501
N	1.14159	8.35886
O	6.02542	7.99985
P	12.27965	7.45208
Q	3.02807	5.72547
R	8.28648	5.59139
S	2.78246	5.45739
T	5.03536	5.01225

2.3 Dual DEA Models

The basic theory of linear programming states that every linear programming problem (usually called the *primal* problem) has another closely related linear program, called its *dual*. Thus, all the linear programming problems developed in Section 2.1 have duals. These duals play a very important role in DEA.

Consider the output maximizing DEA program for Firm A. Let us call this as the primal problem and write its dual.

It is possible to write the dual of any linear programming problem using certain rules. These rules are available in textbooks on linear programming, such as Taha (1997). Following the definitions and rules described in Taha (1997), let us first write the primal problem in standard form as follows.[3]

[3] Please note that the standard form of the primal model is written only in case of the procedure suggested by Taha (1997) for writing the dual of a linear program. Another procedure is to skip this step, and directly write the dual (2.15).

$$\max 1.8v_{VA,A}$$
$$\text{subject to}$$
$$8.6u_{CAP,A} + 1.8u_{EMP,A} = 1$$
$$1.8v_{VA,A} - \left(8.6u_{CAP,A} + 1.8u_{EMP,A}\right) + p_A = 0$$
$$0.2v_{VA,A} - \left(2.2u_{CAP,A} + 1.7u_{EMP,A}\right) + p_B = 0$$
$$2.8v_{VA,A} - \left(15.6u_{CAP,A} + 2.6u_{EMP,A}\right) + p_C = 0$$
$$4.1v_{VA,A} - \left(31.6u_{CAP,A} + 12.3u_{EMP,A}\right) + p_D = 0$$
$$v_{VA,A}, u_{CAP,A}, u_{EMP,A}, p_A, p_B, p_C, p_D \geq 0 \tag{2.14}$$

Let q be the dual variable corresponding to the equality constraint that normalizes the weighted sum of inputs. Let l be the dual variable corresponding to the other inequality constraints of the primal. The dual can be written as follows.

$$\min \theta_A$$
$$\text{such that}$$
$$1.8\lambda_{AA} + 0.2\lambda_{BA} + 2.8\lambda_{CA} + 4.1\lambda_{DA} \geq 1.8$$
$$8.6\theta_A - 8.6\lambda_{AA} - 2.2\lambda_{BA} - 15.6\lambda_{CA} - 31.6\lambda_{DA} \geq 0$$
$$1.8\theta_A - 1.8\lambda_{AA} - 1.7\lambda_{BA} - 2.6\lambda_{CA} - 12.3\lambda_{DA} \geq 0$$
$$\lambda_{AA}, \lambda_{BA}, \lambda_{CA}, \lambda_{DA} \geq 0$$
$$q_A \text{ unrestricted} \tag{2.15}$$

Similar to the notations of the primal, the first subscript of the dual variables refers to all the DMUs, while the second denotes the reference DMU. For convenience, the comma is omitted.

For completeness, the primal (with e constraints) in standard form is also given below.[4]

[4] Please note that the above standard form of the primal is written only in case of the procedure suggested by Taha (1997) for writing the dual of a linear program. Another procedure is to skip this step.

$$
\begin{aligned}
&\max 1.8v_{VA,A} \\
&\text{subject to} \\
&8.6u_{CAP,A} + 1.8u_{EMP,A} = 1 \\
&1.8v_{VA,A} - \left(8.6u_{CAP,A} + 1.8u_{EMP,A}\right) + p_A = 0 \\
&0.2v_{VA,A} - \left(2.2u_{CAP,A} + 1.7u_{EMP,A}\right) + p_B = 0 \\
&2.8v_{VA,A} - \left(15.6u_{CAP,A} + 2.6u_{EMP,A}\right) + p_C = 0 \\
&4.1v_{VA,A} - \left(31.6u_{CAP,A} + 12.3u_{EMP,A}\right) + p_D = 0 \\
&v_{VA,A} - p_{VA} = \varepsilon \\
&u_{CAP,A} - p_{CAP} = \varepsilon \\
&u_{EMP,A} - p_{EMP} = \varepsilon \\
&p_{VA}, p_{CAP}, p_{EMP}, p_A, p_B, p_C, p_D \geq 0
\end{aligned}
\tag{2.16}
$$

Let s and t represent the dual variables corresponding to the e constraints of inputs and outputs, respectively. The dual can be written as follows.

$$
\begin{aligned}
&\min \theta_A - \varepsilon\left(t_{VA,A} + s_{CAP,A} + s_{EMP,A}\right) \\
&\text{such that} \\
&1.8\lambda_{AA} + 0.2\lambda_{BA} + 2.8\lambda_{CA} + 4.1\lambda_{DA} - t_{VA,A} = 1.8 \\
&8.6\theta_A - 8.6\lambda_{AA} - 2.2\lambda_{BA} - 15.6\lambda_{CA} - 31.6\lambda_{DA} - s_{CAP,A} = 0 \\
&1.8\theta_A - 1.8\lambda_{AA} - 1.7\lambda_{BA} - 2.6\lambda_{CA} - 12.3\lambda_{DA} - s_{EMP,A} = 0 \\
&\lambda_{AA}, \lambda_{BA}, \lambda_{CA}, \lambda_{DA}, t_{VA,A}, s_{CAP,A}, s_{EMP,A} \geq 0 \\
&q_A \text{ unrestricted}
\end{aligned}
\tag{2.17}
$$

We shall mostly use the former version of the dual (without the e constraints in the primal). However, our conclusions can be generalized for the other dual too.

2.3.1 Comparing Primal and Dual

Using the basic theory of linear programming, the following observations can be made.

(*a*) As the optimal values of primal and dual objective functions are equal, q_A represents the efficiency of Firm A.
(*b*) The number of constraints of the primal depends upon the number of DMUs, while the number of constraints of the dual depends upon the number of inputs and outputs.
(*c*) The computational efficiency of LP codes depends to a greater extent upon the number of constraints than on the number of variables. In a typical DEA exercise, about 5 inputs and 5 outputs are considered, while the number of units being compared is much larger (of the order of hundreds or even thousands). Hence, the dual formulation is computationally more efficient than the primal.

2.3.2 Interpreting the Dual

Though we have derived the dual mathematically, rather than intuitively, from the primal, we can interpret the dual (2.15) intuitively. First of all, let us observe that while the primal provided optimal weights to inputs and outputs, the dual provides weights to the DMUs (l).

The first dual constraint is the following.

$$1.8\lambda_{AA} + 0.2\lambda_{BA} + 2.8\lambda_{CA} + 4.1\lambda_{DA} \geq 1.8$$

The left-hand side (LHS) of this constraint is the weighted sum of the outputs of all the firms; the right-hand side (RHS) is the output of the reference firm (A here). This constraint states that the dual variables l should be chosen such that the weighted combination of all the outputs of all the firms should be at least equal to the output of the reference firm. As we shall see later, if the firm is efficient, the strict equality will hold, with no slack in the constraint. We shall also see later that for an inefficient firm, the weights are actually the weights to be assigned to their *peers*.

Let us now consider the second constraint of the dual.

$$8.6\lambda_{AA} - 2.2\lambda_{BA} - 15.6\lambda_{CA} - 31.6\lambda_{DA} \leq 8.6\theta_A$$

This constraint corresponds to the capital input. It says that the weighted combination of the capital inputs of all the firms cannot be more than the capital input for the reference firm multiplied by its efficiency. This can be proved using the complementary slackness conditions of linear programming (see Taha 1997) that, at optimality, for an efficient firm, q^* is unity; the variable q^* refers to optimal values. The constraint becomes strict equality with zero slacks (Cooper et al. 2000).

We know that Firm A is an efficient firm. We already know that q_A^* is equal to unity, and the optimal values of slack variables of inputs and outputs are zero. Therefore, the constraints in Equation (2.15) are reduced to,

$$\begin{aligned}
&\theta_A^* = 1\\
&\text{and}\\
&1.8\lambda_{AA}^* = 1.8\\
&8.6\lambda_{AA}^* = 8.6\theta_A^*\\
&1.8\lambda_{AA}^* = 1.8\theta_A^*
\end{aligned} \tag{2.18}$$

Thus,

$$l_{AA}^* = 1$$

in this case. The other ls are equal to zero.

Let us now consider Firm B. The dual DEA program for this firm is the following.

$$\begin{aligned}
&\min \theta_B\\
&\text{such that}\\
&1.8\lambda_{AB} + 0.2\lambda_{BB} + 2.8\lambda_{CB} + 4.1\lambda_{DB} \geq 0.2\\
&8.6\lambda_{AB} + 2.2\lambda_{BB} + 15.6\lambda_{CB} + 31.6\lambda_{DB} \leq 2.2\theta_B\\
&1.8\lambda_{AB} + 1.7\lambda_{BB} + 2.6\lambda_{CB} + 12.3\lambda_{DB} \geq 1.7\theta_B\\
&\lambda_{AB}, \lambda_{BB}, \lambda_{CB}, \lambda_{DB} \geq 0\\
&\theta_B \text{ unrestricted}
\end{aligned} \tag{2.19}$$

We know that this firm is inefficient, and that firm A is its peer. If the LP given above is solved for firm B, the following solution is obtained. (For details of the notations of points A, B, ..., E, please see Figure 1.1.)

$$\theta_B^* = \frac{\text{OE}}{\text{OB}} = 0.434$$

and

$$1.8\lambda_{AB}^* = 0.2$$
$$8.6\lambda_{AB}^* = 2.2\theta_B^*$$
$$1.8\lambda_{AB}^* < 1.7\theta_B^*$$

Note that

$$l_{AB}^* = \frac{0.2}{1.8} = \frac{1}{9}$$

is positive in this case as Firm A is the only peer for Firm B. Other ls are zero in value. The value of q_B can also be obtained from the above equalities.

Thus, mathematically, peers for an inefficient DMU can be identified by solving the dual DEA program. Peers are those efficient DMUs that have positive ls in the optimal solution of the dual DEA program.

Input target for capital input can be obtained as

$$2.2q_B = 0.95$$

Note that the constraint representing capital input,

$$8.6\,l_{AB}^* = 2.2q_B^*$$

is satisfied in equation form; however, there is a slack in the constraint representing employees input

$$1.8\,l_{AB}^* < 1.7q_B^*$$

This is because point E (in Figure 1.1), which represents the best achievable performance for Firm B, corresponds to the same capital input as Firm A but has a higher input of employees. This slack means that though the point E is efficient, there is still scope for reducing the employees input.

In DEA, a DMU is considered efficient if and only if $q^* = 1$ and all slacks are zero; otherwise it is inefficient. However, a DMU may be inefficient in various ways. If $q^* = 1$ at the optimal solution but some of the slack variables do not equal zero, then there exists a combination of other units which does not dominate the current output vector of the reference DMU, but uses less resources. As discussed in Chapter 1, these DMUs may be termed *weakly efficient*. Thus, the firm represented by point E in Figure 1.1 is weakly efficient, while Firms A and B are *strongly efficient* firms.

The value of the slack at point E is,

$$s^*_{EMP,\,B} = 1.7q^*_B - 1.8\ \ l^*_{AB} = 0.54$$

We can now understand why the LHS of the constraints corresponding to inputs are multiplied by the efficiency score q, as can be derived from Equation 2.15: they represent the best achievable performance. For example, for Firm B, $2.2q^*_B$ represents the best achievable value (i.e., input target) for capital employed, while $1.7q^*_B$ represents the best achievable value for employees. Together, these two values represent point E.

Weakly efficient firms can be distinguished from strongly efficient ones using e(infinitesimal) constraints in the dual formulation (2.17). The dual objective function for a firm corresponding to point E is

$$\min \theta_E - \varepsilon s_{EMP,E},$$

which is less than 1 as e and $s^*_{EMP,\,B}$ are positive.

Let us now consider four firms P, Q, R and T that produce the same level of a single output Y, from two inputs X_1 and X_2, shown in Figure 2.1.

Firms P and R are efficient. They represent the *best practice*. This implies that no other firm or no linear combination[5] of firms can be identified which produces the same level of output for less than either or both the inputs.

As we have seen earlier, the dual DEA program for these two firms will indicate zero slacks and unit efficiency. For example, the dual DEA optimal results for Firm P are the following, where

[5] A linear combination of a set of variables is one in which each variable is multiplied by a coefficient and the products summed. For example, it is possible to create a linear combination (Y) of variables X_1, X_2, and X_3 as $Y = 2X_1 + 100X_2 + 0.5X_3$. If the formula for the new variable contains functions such as square roots or logarithms, then the new variable is not a linear combination of the other variables.

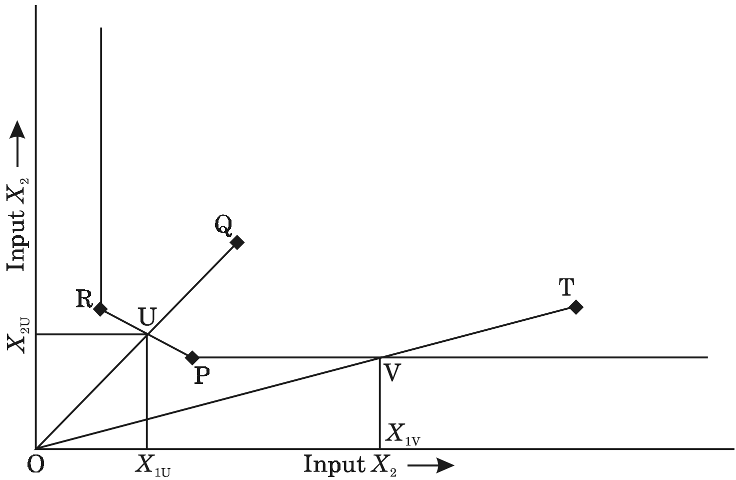


Figure 2.1 Efficiency frontier of four firms consuming two inputs

$l_P^* = q_P^* = 1$. For convenience, the subscript representing the reference firm is suppressed.

$$\theta_P^* = 1$$

and

$$\begin{aligned} X_{1P}^* \lambda_P^* &= X_{1P}^* \theta_P^* \\ X_{2P}^* \lambda_P^* &= X_{2P}^* \theta_P^* \\ Y_P \lambda_P^* &= Y_P \end{aligned} \tag{2.20}$$

Note that both the efficient firms (Firms P and R) are peers for the inefficient Firm Q. Hence, l_P^* and l_R^* are positive. For this firm, the following equalities hold as obtained from the optimal DEA dual results (2.21).

$$\theta_Q^* = \frac{OU}{OQ} \leq 1$$

and

$$\begin{aligned} X_{1P}^* \lambda_P^* + X_{1R}^* \lambda_R^* &= X_{1Q}^* \theta_Q^* \\ X_{2P}^* \lambda_P^* + X_{2R}^* \lambda_R^* &= X_{2Q}^* \theta_Q^* \\ Y_P \lambda_P^* + Y_R \lambda_R^* &= Y_Q \end{aligned} \tag{2.21}$$

Note that RHSs of the above equations represent the best achievable target performance (corresponding to point U) for Firm B. Also, ($X^*_{1Q}q^*_Q = X^*_{1U}$; $X^*_{2Q}q^*_Q = X^*_{2U}$), the hypothetical firm U is actually a linear combination of the best practice firms P and R, where the weights for the linear combination are the ls obtained from the dual DEA program.

For completeness, the dual DEA program corresponding to input minimizing multiplier model for Firm A is given below.

$$
\begin{aligned}
&\min \phi_A \\
&\text{such that} \\
&1.8\mu_{AA} + 0.2\mu_{BA} + 2.8\mu_{CA} + 4.1\mu_{DA} \geq 1.8\phi_A \\
&8.6\mu_{AA} - 2.2\mu_{BA} - 15.6\mu_{CA} - 31.6\mu_{DA} \leq 8.6 \\
&1.8\mu_{AA} - 1.7\mu_{BA} - 2.6\mu_{CA} - 12.3\mu_{DA} \geq 1.8 \\
&\mu_{AA}, \mu_{BA}, \mu_{CA}, \mu_{DA} \geq 0 \\
&\phi_A \text{ unrestricted}
\end{aligned}
\tag{2.22}
$$

Before closing this section, it is important to compare the optimal values of the objective functions q_A and f_A. Note that q_A is the dual objective function for Firm A corresponding to the output maximizing primal, and f_A is the dual objective function for Firm A corresponding to the input minimizing primal. At the optimal solution, the optimal objective function values of the primal and the dual are equal. Hence, by comparing the optimal objective function values of the output maximizing and the input minimizing primals, it is clear that the optimal value of q_A is the reciprocal of the optimal value of f_A.

2.3.3 Two-stage Optimization Procedure

We have seen in Section 2.3 that the use of infinitesimals (e) will distinguish weakly efficient DMUs from strongly efficient ones. As discussed in that section, the dual formulation (with e constraints) to obtain the efficiency of Firm A is the following (Equation 2.17, Section 2.3).

$$\min \theta_A - \varepsilon\left(t_{VA,A} + s_{CAP,A} + s_{EMP,A}\right)$$
such that
$$1.8\lambda_{AA} + 0.2\lambda_{BA} + 2.8\lambda_{CA} + 4.1\lambda_{DA} - t_{VA,A} = 1.8$$
$$8.6\theta_A - 8.6\lambda_{AA} - 2.2\lambda_{BA} - 15.6\lambda_{CA} - 31.6\lambda_{DA} - s_{CAP,A} = 0$$
$$1.8\theta_A - 1.8\lambda_{AA} - 1.7\lambda_{BA} - 2.6\lambda_{CA} - 12.3\lambda_{DA} - s_{EMP,A} = 0$$
$$\lambda_{AA}, \lambda_{BA}, \lambda_{CA}, \lambda_{DA}, t_{VA,A}, s_{CAP,A}, s_{EMP,A} \geq 0$$
$$\theta_A \text{ unrestricted}$$

However, the numeric values for infinitesimals (*e*'s) in actual computations should be chosen to be much smaller than other input and output values so that they will not affect optimization. This is often troublesome.[6] A two-stage optimization procedure (Ali and Seiford 1993; Joro et al. 1998) has been suggested to avoid using the infinitesimals (*e*) in DEA computations.

The two-stage optimization procedure corresponding to the dual formulation (Equation 2.23) for Firm A can be written as follows.

$$\text{lex min}\left\{\theta_A, \left(t_{VA,A} + s_{CAP,A} + s_{EMP,A}\right)\right\}$$
such that
$$1.8\lambda_{AA} + 0.2\lambda_{BA} + 2.8\lambda_{CA} + 4.1\lambda_{DA} - t_{VA,A} = 1.8$$
$$8.6\theta_A - 8.6\lambda_{AA} - 2.2\lambda_{BA} - 15.6\lambda_{CA} - 31.6\lambda_{DA} - s_{CAP,A} = 0$$
$$1.8\theta_A - 1.8\lambda_{AA} - 1.7\lambda_{BA} - 2.6\lambda_{CA} - 12.3\lambda_{DA} - s_{EMP,A} = 0$$
$$\lambda_{AA}, \lambda_{BA}, \lambda_{CA}, \lambda_{DA}, t_{VA,A}, s_{CAP,A}, s_{EMP,A} \geq 0$$
$$\theta_A \text{ unrestricted} \tag{2.23}$$

Here 'lex min' means that the objective function q is minimized first (Stage 1); if the solution is not unique, the second objective function is minimized lexicographically, subject to the additional constraint, that q ³ q^* (where q^* is the optimal value of q) (Stage 2). In case the optimal solution of the first objective function is unique, the second optimization is not needed.

[6] See Chapter 5 for further discussion on the computational features of DEA.

The optimization models for Stages 1 and 2 are given below.

Stage 1

$$
\begin{aligned}
&\min \theta_A \\
&\text{such that} \\
&1.8\lambda_{AA} + 0.2\lambda_{BA} + 2.8\lambda_{CA} + 4.1\lambda_{DA} - t_{VA,A} = 1.8 \\
&8.6\theta_A - 8.6\lambda_{AA} - 2.2\lambda_{BA} - 15.6\lambda_{CA} - 31.6\lambda_{DA} - s_{CAP,A} = 0 \\
&1.8\theta_A - 1.8\lambda_{AA} - 1.7\lambda_{BA} - 2.6\lambda_{CA} - 12.3\lambda_{DA} - s_{EMP,A} = 0 \\
&\lambda_{AA}, \lambda_{BA}, \lambda_{CA}, \lambda_{DA}, t_{VA,A}, s_{CAP,A}, s_{EMP,A} \geq 0 \\
&\theta_A \text{ unrestricted}
\end{aligned} \tag{2.24}
$$

Note that this Stage 1 model is the same as the DEA model with no e constraints.

Stage 2

$$
\begin{aligned}
&\min \left(t_{VA,A} + s_{CAP,A} + s_{EMP,A} \right) \\
&\text{such that} \\
&1.8\lambda_{AA} + 0.2\lambda_{BA} + 2.8\lambda_{CA} + 4.1\lambda_{DA} - t_{VA,A} = 1.8 \\
&8.6\theta_A - 8.6\lambda_{AA} - 2.2\lambda_{BA} - 15.6\lambda_{CA} - 31.6\lambda_{DA} - s_{CAP,A} = 0 \\
&1.8\theta_A - 1.8\lambda_{AA} - 1.7\lambda_{BA} - 2.6\lambda_{CA} - 12.3\lambda_{DA} - s_{EMP,A} = 0 \\
&\theta_A \geq \theta^* \\
&\lambda_{AA}, \lambda_{BA}, \lambda_{CA}, \lambda_{DA}, t_{VA,A}, s_{CAP,A}, s_{EMP,A} \geq 0 \\
&\theta_A \text{ unrestricted}
\end{aligned} \tag{2.25}
$$

where q^* is the optimal value of q_A in stage 1.

2.4 Multiplier and Envelopment DEA Programs

We know that the dual of a dual is primal. Hence, the terms *primal DEA program* and *dual DEA program* are relative. New terms are being increasingly used to represent DEA formulations.

The DEA programs involving weights of inputs and outputs (u and v) are called *Multiplier DEA Programs.* Those involving weights of firms (q and l) are called *Envelopment DEA Programs.*

Output maximizing and input minimizing multiplier versions of DEA programs have been discussed in Section 2.1. A general *envelopment DEA program* corresponding to the output maximizing multiplier model can be written as follows.

$$
\begin{aligned}
&\min \theta_m \\
&\text{such that} \\
&\sum_{n=1}^{N} y_{jn}\lambda_n \geq y_{jm}; \quad j = 1, 2, \mathrm{K}, J \\
&\sum_{n=1}^{N} x_{in}\lambda_n \leq \theta_m x_{im}; \quad i = 1, 2, \mathrm{K}, I \\
&\lambda_n \geq 0; \quad n = 1, 2, \mathrm{K}, N \\
&\theta_m \text{ unrestricted (free)}
\end{aligned}
\tag{2.26}
$$

This model can also be succinctly represented using matrix notation as follows.

$$
\begin{aligned}
&\min_{\theta,\lambda} \theta_m \\
&\text{such that} \\
&Y\lambda \geq Y_m \\
&X\lambda \leq \theta X_m \\
&\lambda \geq 0; \quad \theta_m \text{ free}
\end{aligned}
\tag{2.27}
$$

Similarly, the envelopment DEA program corresponding to the input minimizing multiplier model is given below in its general matrix form.

$$
\begin{aligned}
&\max_{\phi,\mu} \phi_m \\
&\text{such that} \\
&Y\mu \geq \phi_m Y_m \\
&X\mu \leq X_m \\
&\mu \geq 0; \quad \phi_m \text{ free}
\end{aligned}
\tag{2.28}
$$

2.5 Input and Output Oriented Envelopment DEA Programs

Let us study the two envelopment versions, one involving *q* and the other involving *f*. The version involving *q* aims to produce the observed outputs with minimum inputs. That is why inputs are multiplied by efficiency, according to its constraint rules. Because of this characteristic, this version is often referred to as an *input oriented* envelopment DEA program. The other version involving *f* is referred to as an *output oriented* envelopment DEA program as it aims to maximize output production, subject to the given resource level.

Note that the dual of the output maximizing multiplier program is the input oriented envelopment program. Similarly, the dual of the input minimizing multiplier program is the output oriented envelopment program.

Let us examine the behaviour of the input and output oriented envelopment DEA programs more closely, using data regarding the Firms A, B, C, and D. For simplicity, let us consider only one input (capital employed) and only one output (value added).

When the values are plotted in a graph (see Figure 2.2), we observe that Firm A has the maximum output (value added) for a given input (capital). Hence, Firm A is the most efficient, and acts as a peer for all other (inefficient) firms. If we can draw a line

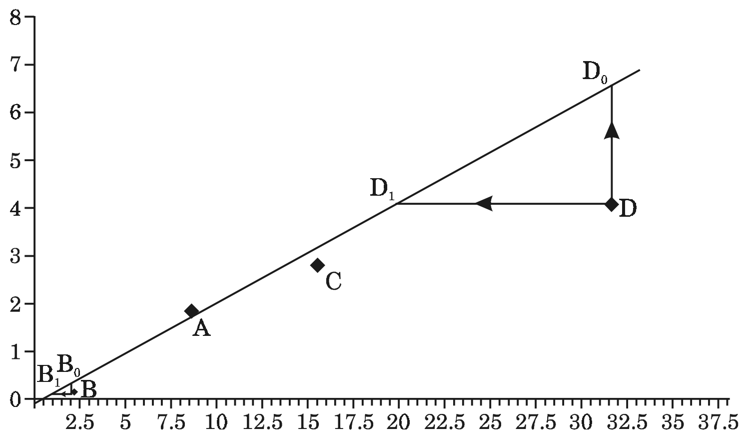


Figure 2.2 Behaviour of input and output oriented envelopment DEA programs

joining the origin to A, all other firms are found to the left of this line.

The input oriented envelopment program used to estimate the efficiency of Firm B is the following.

$$\min \theta_B$$

such that

$$1.8\lambda_{AB} + 0.2\lambda_{BB} + 2.8\lambda_{CB} + 4.1\lambda_{DB} \geq 0.2 \qquad \text{(for value added)}$$

$$8.6\lambda_{AB} + 2.2\lambda_{BB} + 15.6\lambda_{CB} + 31.6\lambda_{DB} \leq 2.2\theta_B \quad \text{(for capital)}$$

$$\lambda_{AB}, \lambda_{BB}, \lambda_{CB}, \lambda_{DB} \geq 0$$

q_B unrestricted (2.29)

Consider its solution. Since Firm A is its peer, zero weights are assigned to Firms C and D (i.e., no other firm influences the performance of Firm B). Hence,

$$\lambda^*_{AB} \neq 0 \quad \text{and} \quad \lambda^*_{BB} = \lambda^*_{CB} = \lambda^*_{DB} = 0$$

$$\lambda^*_{AB} = \frac{0.2}{1.8} = \frac{1}{9}$$

$$\theta^*_B = \frac{8.6 \times \lambda^*_{AB}}{2.2} = 0.434$$

Given q^*_B, the best achievable capital input for Firm B is

2.2 ´ 0.434 = 0.96.

The point is shown by B_1 in Figure 2.2. Thus, the constraint on capital input gives the target for capital input. Therefore, the input oriented envelopment DEA program *projects* the Firm B from *right to left*, as shown by the arrows in Figure 2.2.

Let us study the behaviour of the output oriented envelopment program for the Firm B.

$$\max \phi_B$$

such that

$$8.6\mu_{AB} + 2.2\mu_{BB} + 15.6\mu_{CB} + 31.6\mu_{DB} \leq 2.2 \quad \text{(for capital)}$$

$$1.8\mu_{AB} + 0.2\mu_{BB} + 2.8\mu_{CB} + 4.1\mu_{DB} \leq 0.2\phi_B \quad \text{(for value added)}$$

$$\mu_{AB}, \mu_{BB}, \mu_{CB}, \mu_{DB} \geq 0$$

f_B unrestricted (2.30)

As before, we have the following results.

$$\mu_{AB}^* \neq 0 \text{ and } \mu_{BB}^* = \mu_{CB}^* = \mu_{DB}^* = 0$$

$$\mu_{AB}^* = \frac{2.2}{8.6} = 0.256$$

$$\phi_B^* = \frac{1.8 \times \mu_{AB}^*}{0.2} = 2.30$$

The best achievable performance, using the constraint on output, is $0.2 \times \phi_B^* = 0.46$. This corresponds to the point B_0 in Figure 2.2. Thus, the output oriented envelopment DEA program *projects* the Firm B from *bottom to top*.

The projections can be seen more clearly for the Firm D. The calculations corresponding to this firm are left as an exercise for students.

Further information on this topic can be obtained from Seiford and Thrall (1990).

2.6 Relationships among Different DEA Formulations[7]

So far, we have studied four different DEA programs.

(*a*) Output maximizing multiplier program
(*b*) Input minimizing multiplier program
(*c*) Input oriented envelopment program, and
(*d*) Output oriented envelopment program.

The general matrix representations of the four programs are given below.

[7] This section requires knowledge of matrix algebra, and can be skipped without any loss of continuity.

Multiplier Versions *Output Maximizing*	*Envelopment Versions* *Input Oriented*
$\max_{U,V} z = V_m^T Y_m$ such that $U_m^T X_m = 1$ $V_m^T Y - U_m^T X \leq 0$ $V_m^T, U_m^T \geq 0$ (or $V_m^T, U_m^T \geq \varepsilon$)	$\min_{\theta,\lambda} \theta_m$ such that $Y\lambda \geq Y_m$ $X\lambda \leq \theta_m X_m$ $\lambda \geq 0$; θ_m free
Input Minimizing	*Output Oriented*
$\min_{U',V'} z' = U_m'^T X_m$ such that $V_m'^T Y_m = 1$ $V_m'^T Y - U_m'^T X \leq 0$ $V_m'^T, U_m'^T \geq 0$ (or $V_m'^T, U_m'^T \geq \varepsilon$)	$\max_{\phi,\mu} \phi_m$ such that $Y\mu \geq \phi_m Y_m$ $X\mu \leq X_m$ $\mu \geq 0$; ϕ_m free

Using the theory of duality, we can make certain observations regarding the relationships among the optimal values of these programs. See Seiford and Thrall (1990) for rigorous treatments and proofs of the relationships. In the next few paragraphs, we present the relationships in a more qualitative manner.

It has been explained at the beginning of this chapter (Section 2.1) that

$$z^* = \frac{1}{z'^*}.$$

This is because z and z' represent the denominator and numerator of the equation defining efficiency.

Consider the first constraint of the output maximizing multiplier version of the program. Divide both sides of this equation by z^*. Note that 0 £ z^* £ 1, because efficiency is restricted to be between zero and unity in the corresponding multiplier model. Also, assume that z^* £ 0. We will not use the asterisk in further discussions, but deal only with optimal values.

$$
\begin{aligned}
& U_m^T X_m = 1 \\
& \Rightarrow \frac{U_m^T X_m}{z} = \frac{1}{z} \\
& \Rightarrow \left(\frac{U_m^T}{z} \right) X_m = \frac{1}{z} \\
& \Rightarrow \left(\frac{U_m^T}{z} \right) X_m = z'
\end{aligned}
\tag{2.31}
$$

Comparing the last equation with the objective function of the input minimizing multiplier version, we get,

$$U'^* = \left(\frac{1}{z^*} \right) U^* \tag{2.32}$$

Using a similar argument, we can show

$$V''^* = \left(\frac{1}{z^*} \right) V^* \tag{2.33}$$

Let (q^*, l^*) and (f^*, m^*) be the optimal solutions for the input oriented and output oriented DEA envelopment programs respectively (with the subscript representing the reference firm m suppressed). From analogies with their multiplier versions, we have

$$\phi^* = \frac{1}{\theta^*} \tag{2.34}$$

Divide the two constraints of the input oriented envelopment version of the program by q_m^*. Note that 0 £ q_m^* £ 1 by definition. Further, assume that q_m^* ¹ 0. As before, though asterisks are not used below, let us deal with optimal values only. We have,

$$\frac{Y\lambda}{\theta_m} \geq \frac{Y_m}{\theta_m} \Rightarrow Y\left(\frac{\lambda}{\theta_m} \right) \geq \phi_m Y_m \tag{2.35}$$

$$X\left(\frac{\lambda}{\theta_m} \right) \leq \frac{\theta_m}{\theta_m} X_m \Rightarrow X\left(\frac{\lambda}{\theta_m} \right) \leq X_m \tag{2.36}$$

Comparing the above two equations with the two constraints of output oriented envelopment version of the program, we have

$$\mu^* = \left(\frac{1}{\theta^*}\right)\lambda^* \tag{2.37}$$

We can easily verify this using the numerical calculations we have carried out for Firm B.

$$\frac{1}{\theta^*} = \frac{1}{0.434} = 2.3 = \phi^*$$

$$\left(\frac{1}{\theta^*}\right)\lambda^* = \left(\frac{1}{0.434}\right)\times\left(\frac{1}{9}\right) = 0.256 = \mu^*$$

2.7 Exercises

Some of the questions in this exercise require knowledge of matrix algebra.

1. Write short notes on peers.
2. Say Yes or No with reasons.

   (*a*) The input oriented envelopment DEA problem is the dual of the input minimizing multiplier DEA problem.
   (*b*) The objective function of an input oriented DEA program can take any value.
   (*c*) $\theta^* \frac{1}{\phi^*} = 1$
   (*d*) $u'^* = z'^* u^*$
   (*e*) Output targets for an inefficient DMU can be calculated using the input oriented envelopment program.

3. State the errors in the general envelopment DEA program for a reference firm '*m*' shown below.

$$\begin{aligned}
&\min \omega_m \\
&\text{such that} \\
&Y\gamma \le \omega Y_m \\
&X\gamma \le X_m \\
&\omega, \gamma \ge 0
\end{aligned}$$

4. Interpret the input constraints in an input oriented envelopment DEA program.
5. Compare and contrast the envelopment and multiplier DEA programs.
6. Explain input and output targets and slacks in DEA using a graph for the case of two outputs and one input.
7. Write the input minimizing multiplier DEA program using the following data.

	A	B	C	D	E
Input I	1	2.5	2	6	10
Output O	5	12	11	40	80

Estimate the efficiency of the Firm B for the above data (using hand calculations). Identify its peer in case B is an inefficient firm. Using your results, calculate the optimal objective function value and other parameters for the output oriented DEA program for the same firm.

3

Economies of Scale

In economics, the concept of *production function* specifies the output in an industry for all combinations of inputs. A production function can be depicted on a two-dimensional graph as shown in Figure 3.1. To facilitate depiction on a two-dimensional graph, let us aggregate inputs and outputs, i.e., assume that all the inputs are aggregated into one input, and all outputs are aggregated into one output.

Suppose that a firm consumes inputs amounting to X_1, and produces Y_1 amounts of output. In automated operations, it is possible to consume a certain amount of inputs, and produce more than a proportional amount of output. For example, consider a manufacturer producing shock absorbers. If only a few shock absorbers need to be produced, he may prefer to do so manually. But, if he needs to produce a large amount of shock absorbers, he may prefer automate his process. Hence he will be able to produce a larger amount of output in proportion to the inputs. Therefore, he can consume a larger input X_2, and can produce the output Y_2, which is more than a proportional increase in output, i.e.,

$$\frac{Y_2}{Y_1} > \frac{X_2}{X_1}. \tag{3.1}$$

This concept is termed *Economy of Scale*. Actually, the manufacturer is operating under *Increasing Returns to Scale* as his returns (profits) will increase if he increases his production.

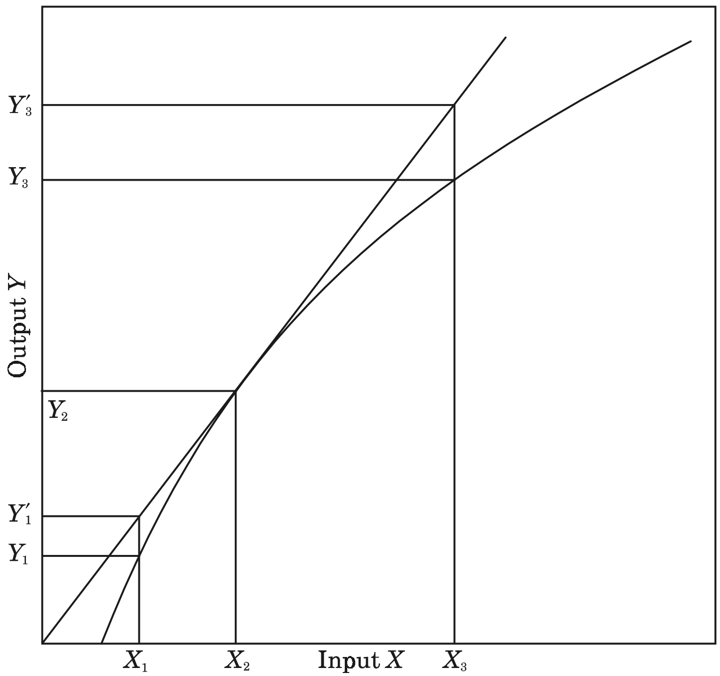


Figure 3.1 Production functions

One can define *Increasing Returns to Scale* (IRS) as a property of a production function such that changing all inputs by the same proportion changes the output by a greater extent than the proportional value.

However, beyond a limit, IRS does not hold. If the manufacturer needs to produce billions of shock absorbers, he might find it difficult to produce that amount because of storage problems and limits on the supply of raw materials. In this case, he is said to be operating under *Decreasing Returns to Scale* (DRS).

Combining the two extremes (IRS and DRS) would necessitate Variable Returns to Scale (VRS). This property signifies that in a production process, the operations will follow IRS or DRS (or CRS—see discussion below) for different ranges of output. The same concept can be extended to areas other than production processes, such as schools, banks, hospitals, and other categories of DMUs.

Note that the IRS changes to DRS at a particular level of production, represented by (X_2, Y_2) in Figure 3.1. At this point a DMU is said to be operating at its *Most Productive Scale Size* (MPSS), because it enjoys the maximum possible economy of scale.

Another variant of economies of scale is *Constant Returns to Scale* (CRS). This property signifies that the manufacturer is able to scale the inputs and outputs linearly without increasing or decreasing efficiency. In such a case, he is able to obtain of output Y_1' by consuming X_1 of input, Y_2 by consuming X_2 and, Y_3' by consuming X_3. This is a significant assumption because may be valid over limited ranges. Hence, if the CRS assumption is employed for a particular case, its use must be justified showing evidences for the existence of CRS.

3.1 Returns to Scale and DEA

It is important to note that the DEA models discussed so far assume that the operations follow constant returns to scale. This represented one of the most limiting factors for the applicability of DEA, at least in the early years. DEA has not received widespread attention for the analysis of production processes because of this limitation. Many economists viewed this assumption as over-restrictive and preferred alternative statistical procedures in spite of the advantages offered by DEA.

Modifications on DEA to handle VRS categories were first described in 1984, when Banker et al. (1984) came up with a simple yet remarkable modification to the CCR DEA models in order to handle variable returns to scale. This modification was suggested by comparing some previous studies on production functions. We will not report the previous studies, and hence will not provide rigorous proofs of the modification, but we shall certainly study the effect of modification intuitively. Those interested in rigorous proof should consult Banker et al. (ibid.).

Before discussing returns to scale properties in the context of DEA, let us complete estimating the parameters (q and l) for the Firms C and D (for the case of one input—capital employed, CAP and one output—value added, VA). Students are advised to check these computations.

for Firm C,

$$l^*_{AC} = 1.56 \quad \text{and} \quad q^*_C = 0.86$$

for Firm D,

$$l^*_{AD} = 2.28 \quad \text{and} \quad q^*_D = 0.62$$

Also, as previously calculated,
for Firm A,

$$l^*_{AA} = 1 \quad \text{and} \quad q^*_A = 1$$

for Firm B,

$$l^*_{AB} = 1/9 \quad \text{and} \quad q^*_B = 0.434.$$

Observe that,

$$l^*_{AA} = 1;\; l^*_{AB} < 1;\; l^*_{AC} > 1 \text{ and } l^*_{AD} > 1.$$

We can identify a relationship between the values of l and the scales of operation of the firms. The performances of the four Firms A, B, C, and D are plotted in Figure 3.2. Note that the scale of operation of Firm B is smaller, and that of Firms C and D is larger, as compared to the efficient Firm A. Accordingly, the weights of l_A for these firms differ. The target for Firm B is scaled

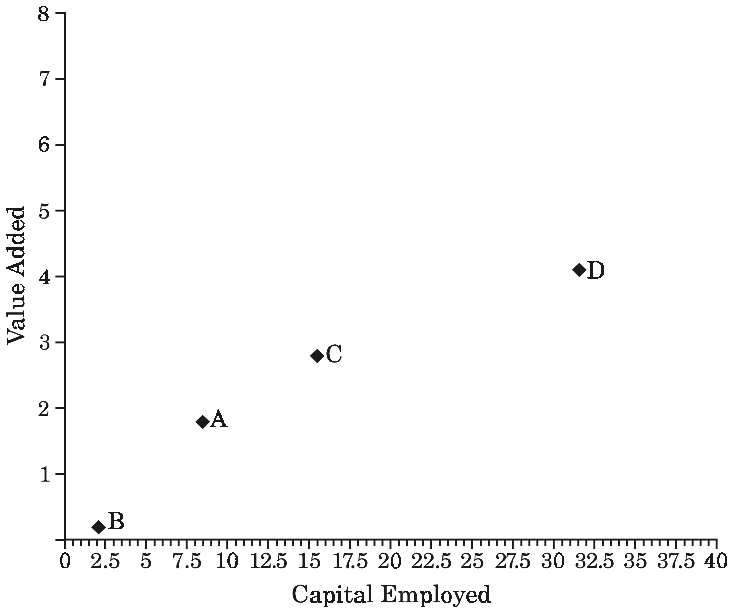


Figure 3.2 Scale of operations of Firms A, B, C, and D

down as depicted by the relation $\lambda^*_{AB} < 1$, while the targets for Firms C and D are scaled up as depicted by the relationships $\lambda^*_{AC} > 1$ and $\lambda^*_{AD} > 1$.

It has already been shown in previous chapter that, when only capital employed is considered as input and value added as output, Firm A displays the highest ratio of input to output. Hence, Firm A is the most efficient and is considered to be operating at the most productive scale size. Firms operating at lower scale sizes (such as Firm B) are said to be operating under IRS because they can achieve greater economies of scale if they increase their volume of operation. Note that as observed earlier,

$$\lambda^*_{AB} < 1.$$

Firms operating at higher scales sizes (such as Firms C and D) are said to be operating under DRS. Again, as observed earlier, note that

$$\lambda^*_{AC} > 1 \quad \text{and} \quad \lambda^*_{AD} > 1$$
$$\text{and also } \lambda^*_{AA} = 1.$$

In other words, a useful test of returns to scale properties of DMUs can be obtained by observing the corresponding values of λ^*.

(*a*) If $\lambda^*_{bp} < 1 \Rightarrow$ Increasing Returns to Scale, IRS
(*b*) If $\lambda^*_{bp} > 1 \Rightarrow$ Decreasing Returns to Scale, DRS

where *bp* denotes the *best practice* DMU. Note that $\lambda^*_{bp} = 1$ for the best practice DMU.

We have considered one input (capital employed, CAP) and only one output (value added, VA). But, in practice, we may need to consider a greater number of inputs and outputs. In such cases, the foregoing conditions will be modified as follows (Ganley and Cubbin 1992).

$$\sum_{n=1}^{N} \lambda_n < 1 \Rightarrow \text{Increasing Returns to Scale, IRS}$$
$$\sum_{n=1}^{N} \lambda_n > 1 \Rightarrow \text{Decreasing Returns to Scale, DRS}$$

Note that, for efficient firms,

$$\sum_{n=1}^{N} \lambda_n = 1.$$

Note also that some firms that are not efficient in the models considered so far may become efficient if we assume variable returns to scale relaxing the assumption of CRS.

Rating firms as efficient/inefficient depends upon the constraints imposed on a CCA DEA program. Suppose we force the condition $\sum_{n=1}^{N} \lambda_n = 1$ in the CCR DEA program. The introduction of this additional constraint ensures that firms operating at different scales are recognized as efficient. Therefore, the envelopment is formed by the multiple convex linear combinations of best practice (incorporating VRS). The constraint $\sum_{n=1}^{N} \lambda_n = 1$ is termed the *convexity constraint* in the mathematics literature.

The VRS frontier for the four firms is shown in Figure 3.3.

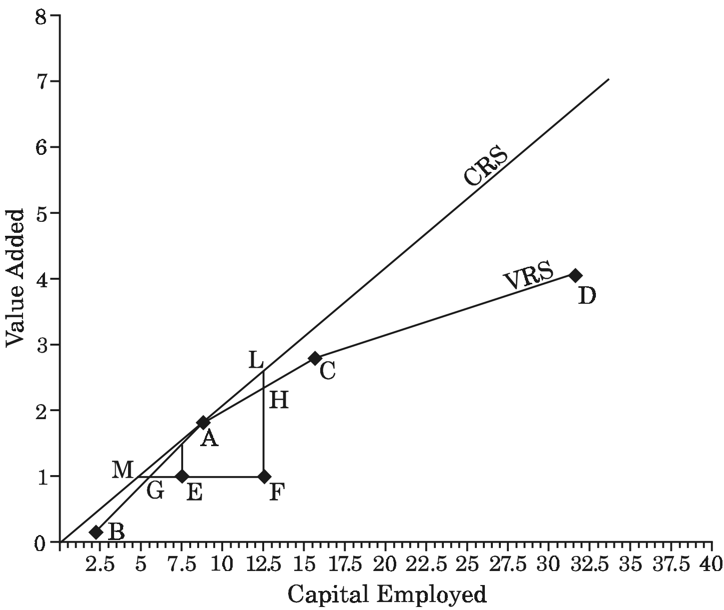


Figure 3.3 CRS and VRS frontiers for the Firms A, B, C, and D

3.1.1 Variable Returns to Scale Envelopment DEA Programs

Thus, the DEA envelopment program for considering variable returns to scale is the following.

$$
\begin{aligned}
&\min_{\theta,\lambda} \theta_m \\
&\text{such that} \\
&Y\lambda \geq Y_m \\
&X\lambda \leq \theta X_m \\
&\sum_{n=1}^{N} \lambda_n = 1 \quad (\text{or, } e^T\lambda = 1, \text{where } e \text{ is a unit vector}) \\
&\lambda \geq 0; \quad \theta_m \text{ free}
\end{aligned} \tag{3.2}
$$

As mentioned earlier, this modification was first suggested by Banker et al. (1984). Hence, the foregoing DEA model is termed the *BCC* (Banker, Charnes and Cooper) *model*. In general, DEA programs incorporating the additional convexity constraint to take into account variable returns to scale are called *BCC DEA models* or VRS DEA models. In contrast, CCR DEA models are also called CRS DEA models.

According to Figure 3.3, all the four firms have been recognized as efficient. They were considered inefficient by the CCR program because of their differences in scale sizes.

Consider another Firm E using CAP = 7.5 to produce VA = 1. It will certainly be designated as inefficient because Firms B and A operate more efficiently, though they are smaller and larger in size, respectively, as compared to E. The VRS efficiency of Firm E is 0.72, with

$$l^*_{AE} = 0.5 = l^*_{BE}.$$

Note that the input oriented as well as output oriented envelopment models will project E on to the Facet AB.

Consider another Firm F. As can be observed from Figure 3.3, it is inefficient, but it is projected onto different facets depending upon which orientation model is used. The input oriented model projects Firm F on the Facet AB on to point G, while the output oriented model projects it at the Facet AC on to point H. Of course, without the convexity constraint

$$\sum_{n=1}^{N} \lambda_n = 1,$$

the CCR envelopment models will project the firm onto the points M and L respectively.

3.1.2 Non-increasing and Non-decreasing Returns to Scale Envelopment DEA Programs

We now know that appending the constraint $\sum_{n=1}^{N} \lambda_n = 1$ has the effect of introducing VRS into the model. Not appending such a constraint has the effect of introducing CRS.

What is the effect of introducing the constraints, $\sum_{n=1}^{N} \lambda_n \leq 1$ or $\sum_{n=1}^{N} \lambda_n \geq 1$?

Suppose, we add the constraint $\sum_{n=1}^{N} \lambda_n \leq 1$. Firm B, which is at IRS, will be chosen as efficient only if $\sum_{n=1}^{N} \lambda_n = 1$ is forced. But, $\sum_{n=1}^{N} \lambda_n \leq 1$ does not force this. Without any convexity constraint, Firm B has the constraint $\sum_{n=1}^{N} \lambda_n < 1$. This requirement is allowed by the condition $\sum_{n=1}^{N} \lambda_n \leq 1$. Hence, Firm B will not be considered efficient.

In contrast, for Firms C and D, $\sum_{n=1}^{N} \lambda_n > 1$, which is not allowed by the condition $\sum_{n=1}^{N} \lambda_n \leq 1$. Hence, the condition $\sum_{n=1}^{N} \lambda_n = 1$ is forced, in their case. Accordingly, they will be considered efficient by the model.

Thus, adding the constraint, $\sum_{n=1}^{N} \lambda_n \leq 1$, has the effect of forcing CRS up to Firm A, and VRS beyond it. Firm B, which is operating under IRS, will be considered inefficient, while Firms C and D, which are operating under DRS, will be considered efficient. Thus the resulting model will be said to be capturing Non-Increasing Returns to Scale (NIRS). The NIRS frontier is shown in Figure 3.4.

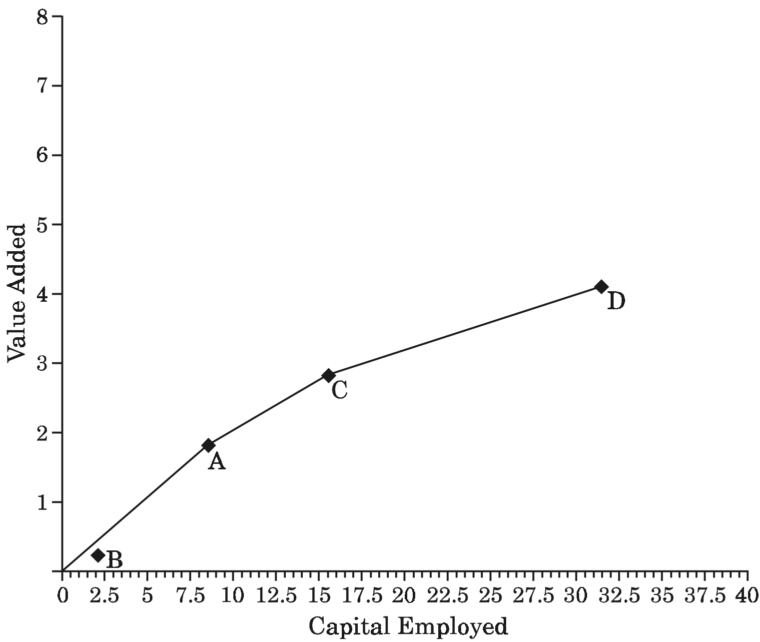


Figure 3.4 NIRS DEA frontier

Using a similar argument, we can prove that the condition $\sum_{n=1}^{N} \lambda_n \geq 1$ will capture Non-Decreasing Returns to Scale (NDRS). If this constraint is introduced, Firm B will be considered efficient, while Firms C and D will be considered inefficient. The NDRS frontier is shown in Figure 3.5.

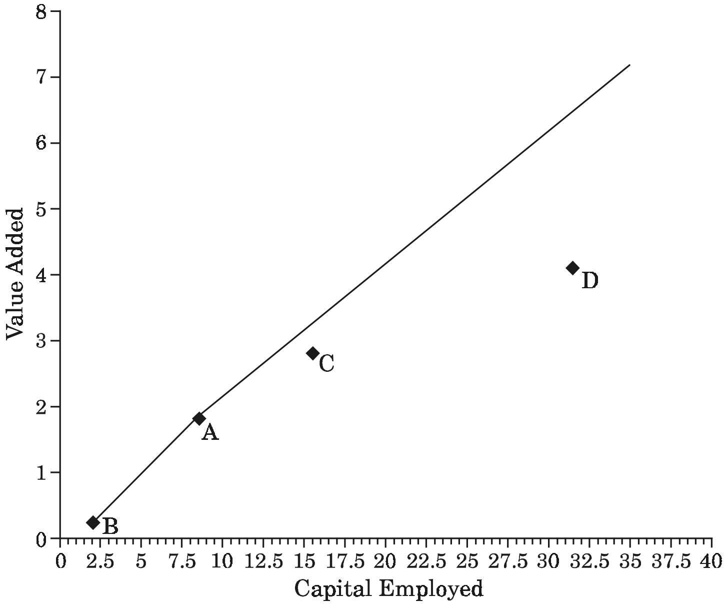


Figure 3.5 NDRS DEA frontier

3.2 Variable Returns to Scale Multiplier DEA Programs

What is the effect of introducing the constraint $\sum_{n=1}^{N} \lambda_n = 1$ on the dual of the envelopment program (i.e., the multiplier program)? Let us write the multiplier DEA program now.

Let us consider the VRS envelopment program for Firm B (the BCC model).

$$
\begin{aligned}
&\min \theta_B \\
&\text{such that} \\
&1.8\lambda_{AB} + 0.2\lambda_{BB} + 2.8\lambda_{CB} + 4.1\lambda_{DB} \geq 0.2 \\
&8.6\lambda_{AB} + 2.2\lambda_{BB} + 15.6\lambda_{CB} + 31.6\lambda_{DB} \leq 2.2\theta_B \\
&1.8\lambda_{AB} + 1.7\lambda_{BB} + 2.6\lambda_{CB} + 12.3\lambda_{DB} \leq 1.7\theta_B \\
&\lambda_{AB} + \lambda_{BB} + \lambda_{CB} + \lambda_{DB} = 1 \\
&\lambda_{AB}, \lambda_{BB}, \lambda_{CB}, \lambda_{DB} \geq 0 \\
&\theta_B \text{ unrestricted}
\end{aligned}
\tag{3.3}
$$

Let v_{0B} be the dual variable corresponding to the convexity constraint (the subscript B refers to the reference DMU). The dual is shown below.

$$\max 0.2v_{VA,B} + v_{0B}$$
subject to
$$2.2u_{CAP,B} + 1.7u_{EMP,B} = 1$$
$$1.8v_{VA,B} - \left(8.6u_{CAP,B} + 1.8u_{EMP,B}\right) + v_{0B} \le 0$$
$$0.2v_{VA,B} - \left(2.2u_{CAP,B} + 1.7u_{EMP,B}\right) + v_{0B} \le 0$$
$$2.8v_{VA,B} - \left(15.6u_{CAP,B} + 2.6u_{EMP,B}\right) + v_{0B} \le 0$$
$$4.1v_{VA,B} - \left(31.6u_{CAP,B} + 12.3u_{EMP,B}\right) + v_{0B} \le 0$$
$$v_{VA,B}, u_{CAP,B}, u_{EMP,B} \ge 0$$
$$v_{0B} \text{ free} \tag{3.4}$$

Thus, the addition of the convexity constraint in the envelopment program results in the introduction of another variable, v_{0B}, in the corresponding multiplier program. Note that v_{0B} is a free variable in the above formulation. Can we interpret the new variable intuitively?

Suppose that the convexity constraint $\sum_{n=1}^{N} \lambda_n = 1$ in the envelopment program is modified to $\sum_{n=1}^{N} \lambda_n \ge 1$. How does the dual change?

Verify that the multiplier program is the same as shown previously, except that v_{0B} is not a free variable now, but that v_{0B} ³ 0. We know that the constraint $\sum_{n=1}^{N} \lambda_n \ge 1$ leads to NDRS. Hence, we can deduce that, if the optimal value of v_0 is positive, then the DMU is characterized by NDRS.

By a similar logic, we can deduce that if v_{0B} £ 0, then the firm is characterized by NIRS.

Note that the CCR model did not have the variable v_{0B}. Hence, we can say that $v_{0B} = 0$ for CRS.

As a summary,

$$
\begin{aligned}
&v_0 \geq 0 \Rightarrow \text{NDRS}\\
&v_0 = 0 \Rightarrow \text{CRS}\\
&v_0 \leq 0 \Rightarrow \text{NIRS}\\
&v_0 \text{ is free for VRS}
\end{aligned}
$$

Combining the above, we have,

$$
\begin{aligned}
&v_0^* > 0 \Rightarrow \text{IRS}\\
&v_0^* = 0 \Rightarrow \text{CRS}\\
&v_0^* < 0 \Rightarrow \text{DRS}
\end{aligned}
$$

Thus, if the optimal value of v_0 is positive, we can conclude that the DMU is operating under IRS.

3.3 Technical and Scale Efficiencies

Given the fact that firms are assigned different efficiencies in case of CRS and VRS assumptions, i.e., using CCR models and BCC models, we can distinguish two different kinds of efficiencies—*Technical* and *Scale Efficiencies*.

The CCR model (without the convexity constraint) estimates the gross efficiency of a DMU. This efficiency comprises technical efficiency and scale efficiency. Technical efficiency describes the efficiency in converting inputs to outputs, while scale efficiency recognizes that economy of scale cannot be attained at all scales of production, and that there is one *most productive scale size*, where the scale efficiency is maximum at 100 per cent.

The BCC model takes into account the variation of efficiency with respect to the scale of operation, and hence measures pure *Technical Efficiency*.

The CRS and VRS frontiers for the four Firms A, B, C and D are shown in Figure 3.6. Note that while only Firm A is assigned 100 per cent efficiency in the case of the CRS assumption, all the firms are considered 100 per cent efficient in case of the VRS assumption. This indicates that the inefficiencies assigned to Firms B, C, and D in case of the CRS assumption are purely due to their scales of operation.

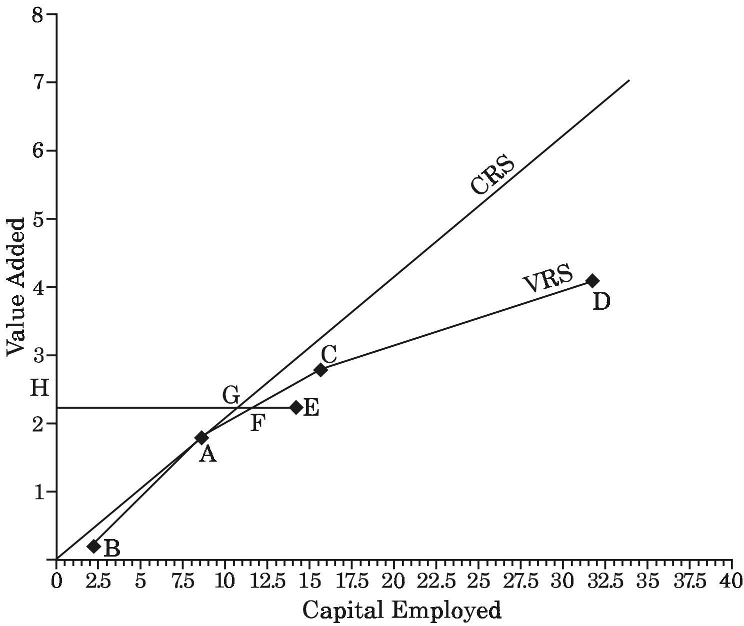


Figure 3.6 CRS and VRS efficient frontiers for the Firms A, B, C and D

Consider another Firm E, with value added (VA) of $2.2 million and capital employed (CAP) of $14 million, as shown in Figure 3.6. Obviously, Firm E is inefficient in case of both CRS and VRS assumptions. It is considered inefficient in case of the CRS assumption because its ratio of VA to CAP is 0.1571, which is smaller than the corresponding ratio for Firm A. It is considered inefficient in case of the VRS assumption because Firms A and C operate at lower and higher scales, respectively, compared to E and have higher VA/CAP ratios. The application of the DEA models show that the CRS efficiency of Firm E is 75.08 per cent while its VRS efficiency is 81.43 per cent.

The VRS efficiency (solved using the BCC model) is given by the following.

$$\text{(VRS efficiency of E)} = \text{(Pure Technical Efficiency)} = HF/HE$$

The CRS efficiency (solved using the CCR model) is given by the following.

(CRS efficiency of E) = (Technical and Scale Efficiency) = HG/HE

In other words, scale efficiency of a DMU can be computed as the ratio of its CRS efficiency to its VRS efficiency. Hence, *Scale Efficiency of Firm E*, caused purely by the fact that E does not operate at the most productive scale size, is given by HG/HF.

Note that,

Technical and Scale Efficiency (CCR efficiency)
= HG/HE
= (HG/HF) ´ (HF/HE)
= Scale Efficiency ´ Technical (VRS) Efficiency (3.5)

Thus, the scale efficiency of Firm E can be obtained as the ratio of its CRS efficiency to its VRS efficiency, which is (75.08/81.43), or 92.2 per cent.

The CRS efficiency of a firm is always less than or equal to the pure technical (VRS) efficiency.

CRS efficiency £ VRS efficiency

The equality holds when the scale efficiency is unity, or when the DMU is operating at the *Most Productive Scale Size* (MPSS). Thus, other things being equal, the VRS technique gives the highest efficiency score, while the CRS technique gives the lowest score.

3.4 Estimation of the Most Productive Scale Size

We have discussed in Section 3.3 that CCR efficiency accounts for scale inefficiency also. That is, CCR efficiency takes into account the fact that DMUs operate at scales that are not their *Most Productive Scale Size* (MPSS). In such a case, if a DMU does not operate at its MPSS, what is its MPSS? That is, if the present scale of operation of a DMU does not lead to 100 per cent scale efficiency, then what is the scale size it should operate at, to achieve 100 per cent scale efficiency?

The identification of MPSS is easy in case of single input/single output. For example, Firm B is not an efficient one under the

CRS assumption, but the MPSS for the firm is given by the size of the efficient Firm A. However, it is not as easy to identify MPSS when dealing with multiple inputs and outputs. Note that, for more than one input and output, there will be more than one DMU that is CCR efficient. Further, it is not easy to check which of the CCR-efficient DMUs will form the MPSS for a given inefficient firm.

Mathematically, the information about MPSS for a CRS-inefficient firm is contained in the weights of its peers (l). Let us study the following relationships.

$$\text{MPSS of VA for Firm B} = \frac{\text{VA for Firm B}}{\sum_{n=1}^{N} \lambda_n}$$

$$= \frac{0.2}{\lambda_{AB}^{*}}$$

$$= \frac{0.2}{(1/9)}$$

$$= 1.8$$

$$= \text{VA for Firm A}$$

$$\text{MPSS of CAP for Firm B} = \theta_B^{*} \times \frac{\text{CAP for Firm B}}{\sum_{n=1}^{N} \lambda_n}$$

$$= \frac{0.434 \times 2.2}{\lambda_{AB}^{*}}$$

$$= \frac{0.9548}{(1/9)}$$

$$= 8.6$$

$$= \text{CAP for Firm A}$$

The same logic can be extended for cases involving greater numbers of inputs and outputs. Banker (1984) has proved that MPSS for a given inefficient firm can be obtained using the following relationship.

$$\left(X_{i,m}^{MPSS}, Y_{j,m}^{MPSS}\right) = \left(\theta_m^* \frac{X_{im}}{\sum_{n=1}^{N} \lambda_{nm}^*}, \frac{Y_{jm}}{\sum_{n=1}^{N} \lambda_{nm}^*}\right) \tag{3.6}$$

where q_m^* and l_{nm}^* are obtained for the reference firm m using the CCR input oriented envelopment DEA model. Note that q_m^* and $\sum_{n=1}^{N} \lambda_{nm}^*$ both equal unity for a CCR-efficient firm (which operates at its MPSS).

Similarly, if the output oriented CCR model is used, then MPSS for a given inefficient DMU can be obtained as follows.

$$\left(X_{i,m}^{MPSS}, Y_{j,m}^{MPSS}\right) = \left(\frac{X_{im}}{\sum_{n=1}^{N} \mu_{nm}^*}, \phi_m^* \frac{Y_{jm}}{\sum_{n=1}^{N} \mu_{nm}^*}\right) \tag{3.7}$$

3.5 Investigating the Returns to Scale Properties of a DMU

The discussion so far in this chapter provides two different methods for investigating the nature of a DMUs returns to scale. The first one, detailed in Section 3.1, uses the sum of the optimal values of all the ls when the CCR envelopment version is solved, considering the DMU in question as the reference DMU. Thus,

if $\sum_{n=1}^{N} \lambda_n^* = 1$, then the reference DMU is expected to exhibit CRS,

if $\sum_{n=1}^{N} \lambda_n^* > 1$, then the reference DMU is expected to exhibit IRS, and if $\sum_{n=1}^{N} \lambda_n^* > 1$, then the reference DMU is expected to exhibit DRS.

The second method, discussed in Section 3.2, uses the value of v_0 when the VRS multiplier version is solved, considering the DMU in question as the reference DMU.

If $v_0^* = 0$, then the reference DMU is expected to exhibit CRS, if $v_0^* > 0$, then the reference DMU is expected to exhibit IRS, and if $v_0^* < 0$, then the reference DMU is expected to exhibit DRS.

However, these tests may fail when the DEA models have alternate optima. Interested students should refer to Seiford and Zhu (1999) for further details.

The third method, usually called the *scale efficiency method* (Fare et al. 1985), is robust against multiple optima (Seiford and Zhu 1999). In this method, three different DEA models are solved, considering the DMU whose returns to scale is to be assessed as the reference DMU.

(*a*) The first model is the CCR DEA model. For example, the input oriented envelopment CCR DEA model does not have any convexity constraint involving $\sum_{n=1}^{N} \lambda$. Let the optimal objective function value be denoted as p.

(*b*) The second model is the BCC DEA model. For example, the input oriented envelopment BCC DEA model has the additional constraint, $\sum_{n=1}^{N} \lambda = 1$. Let the optimal objective function value be denoted as q. We have seen earlier that the ratio p/q is the scale efficiency of the reference DMU. If the scale efficiency is 1, i.e., if $p = q$, then the reference DMU exhibits CRS.

(*c*) Otherwise, if $p \neq q$, a third model, the NIRS DEA model, needs to be solved. For example, the input oriented envelopment NIRS DEA model has the additional constraint,

$\sum_{n=1}^{N} \lambda \leq 1$, as compared to the basic CCR DEA model. Let the optimal objective function value be denoted as r. If $q > r$, then the reference DMU exhibits IRS, and if $q = r$, then the reference DMU exhibits DRS (Fare and Grosskopf 1985; Seiford and Zhu 1999).

Let us use the scale efficiency method to assess the returns to scale properties for Firms B and D.

For Firm B,

$$p = 0.434,$$
$$q = 1.000 \text{ and}$$
$$r = 0.434.$$

Since p ¹ q, we conclude that Firm B does not exhibit CRS. Further, since $q > r$, Firm B exhibits IRS.

For Firm D,

$$p = 0.620,$$
$$q = 1.000 \text{ and}$$
$$r = 1.000.$$

Since p ¹ q, we conclude that Firm D does not exhibit CRS. Further, since $q = r$, Firm D exhibits DRS.

3.6 Exercises

1. Write all the eight DEA programs for estimating efficiencies of the four Firms A, B, C and D.

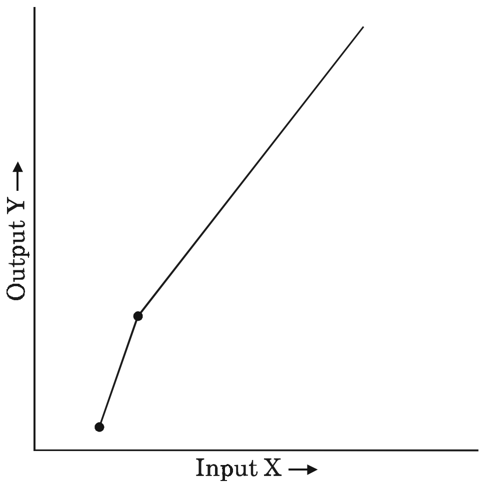
   (*a*) Envelopment: input oriented and output oriented
   (*b*) Multiplier: output maximizing and input minimizing
   (*c*) Returns to scale: constant and variable

Solve the programs using an LP software. Tabulate the results. You may use the results of the previous exercises wherever needed.

2. Solve the problems of the previous question using a DEA software. (See Chapter 5 on a discussion on available DEA software.) Compare the results.
3. A VRS-efficient firm will operate at its MPSS. Comment.
4. Say Yes or No with reasons.

(*a*) The introduction of the constraint å *l* £ 1 to the CRS envelopment version will force the DEA program to assume Non-Increasing Returns to Scale.

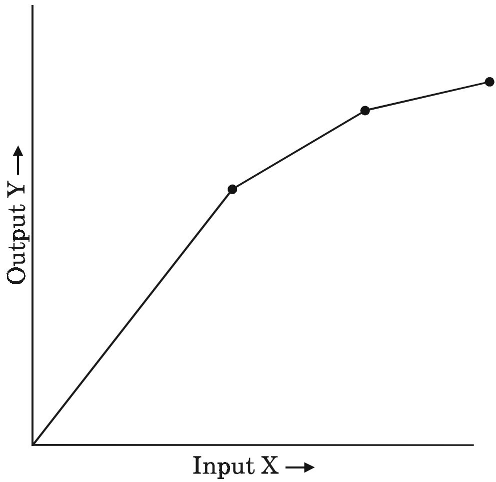
(*b*) The figure below represents firms evaluated using the NDRS assumption.



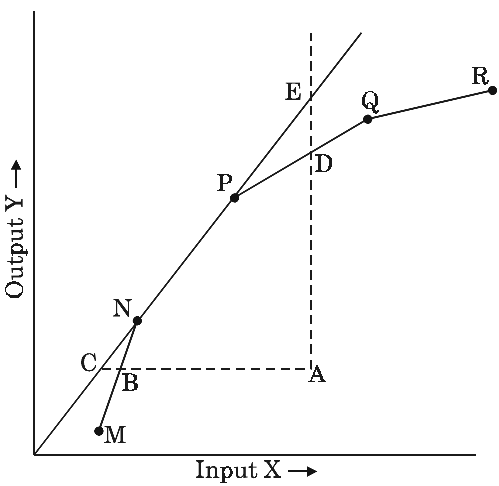
(*c*) The figure in the next page represents firms evaluated using the constraint å *l* ³ 1.

(*d*) Decreasing returns to scale in an output maximizing DEA multiplier program is forced by introducing the variable v_0 in the objective function only such that $v_0 < 0$.

(*e*) For a firm, VRS efficiency £ CRS efficiency.



5. Consider the figure shown below involving six firms, M, N, P, Q, R and A. Say Yes or No with reasons.



(*a*) Firms N and P are the only two efficient firms.
(*b*) Firm A is inefficient.
(*c*) An input oriented CRS DEA program for Firm A will project this firm on to the point D.

(*d*) M operates at its MPSS.
(*e*) Firm P is 100 per cent scale efficient.
(*f*) If one introduces the constraint $\sum \lambda \geq 1$ to the CCR envelopment model, Firm M will be considered efficient.
(*g*) Write the ratios for the CRS, VRS and scale efficiencies of Firm A.

The following questions have been included as practice problems for demonstrating the variety of applications of DEA. Several applications of DEA have been described in Chapter 6, each for of which the data and the DEA results have been presented. Students should consider them also as practice problems. In addition, several data sets for practicing DEA are available on the Internet. See Chapter 5 for more details.

6. The data, choice of inputs and choice of output in this exercise are based on the article, Sueyoshi and Goto (2001). The following data represents, for the year 1993, three inputs (the amount of total fossil fuel generation capacity in Mega Watts (MW); the amount of total fuel consumption in 10^9 kilocalories (kcal); and the number of total employees in fossil fuel plants) and one output (the amount of total generation in Giga Watt-hour [GWh]) pertaining to electric power generation companies in Japan.

   Write all the eight DEA programs (Envelopment: input oriented and output oriented, Multiplier: output maximizing and input minimizing, and Returns to scale: constant and variable) for estimating the efficiency of the power generation company located in Hokkaido.

   Solve the programs using an LP software. Tabulate the results. Is the Hokkaido power generation company efficient? If not, which power generation companies would you recommend the Hokkaido company consider emulating to improve the efficiency of its operation? What are the CRS, VRS and scale efficiencies of this company? What is its most productive scale size?

   Which are the efficient power generation companies in this data set? For each of them, prepare a table showing the inefficient companies for whom the efficient company is a peer.

A data set of Japanese electric power generation companies

Name (DMU)	*Generation Capacity (MW)*	*Fuel Consumption (10^9 kcal)*	*Number of Employees*	*Total Generation (GWh)*
Chubu	18,075	1,53,335	2,472	69,300
Chugoku	6,406	56,194	1,176	25,717
Dengen-kaihatsu	4,655	65,728	825	30,071
Fukui	250	754	59	326
Fukuyama	699	9,704	187	4,339
Hokkaido	3,060	28,850	659	13,120
Hokuriku	2,662	17,127	664	7,648
Jyouban	1,625	15,758	206	6,940
Kansai	18,581	1,14,707	3,132	50,846
Kashima	1,400	12,380	141	5,407
Kimitsu	950	12,745	177	5,533
Kyushu	9,321	57,229	1,480	27,448
Mizushima	543	6,946	106	3,107
Okinawa	1,290	8,134	477	4,018
Ooita	500	7,837	85	3,351
Sakai	150	1,530	63	594
Sakata	700	9,468	86	4,229
Shikoku	3,171	29,611	554	13,113
Sumitomo	463	2,456	97	990
Tobata	781	10,446	104	4,488
Tohoku	7,868	73,705	107	34,228
Tokya	29,254	2,80,478	3,751	1,27,538
Tomakomai	250	1,817	55	790
Toyama	500	6,192	84	2,703
Wakayama	306	4,541	112	1,921

7. The performance of some selected international airlines is given in the following table. Operating costs, nonflight assets and nonpassenger revenue are measured in appropriate money units (US$).

   Write all the eight DEA programs (Envelopment: input oriented and output oriented, Multiplier: output maximizing and input minimizing, and Returns to scale: constant and variable) for estimating the efficiency of Airline 4.

   Solve these programs using an LP software. Tabulate the results. Is Airline 4 efficient? If not, which airlines would you recommend that Airline 4 consider emulating to improve the efficiency of its operation? What are the CRS, VRS and

A data set for selected international airlines

	Inputs			Outputs	
Airline (DMU)	Available ton km	Operating Cost	Non-flight Assets	Revenue Passenger (km)	Non-passenger Revenue
1	10,884	6,730	3,934	26,677	7,688
2	4,603	3,457	2,360	22,112	969
3	12,097	6,779	6,474	52,363	2,001
4	6,587	3,341	3,581	26,504	1,297
5	5,723	3,239	2,003	26,677	697
6	24,099	9,560	6,267	1,24,055	1,266
7	22,793	9,874	4,145	1,22,528	1,404
8	19,080	8,032	3,272	95,011	572
9	5,654	1,878	1,916	19,277	972
10	12,559	8,098	3,310	41,925	3,398
11	5,728	2,481	2,254	27,754	982
12	4,715	1,792	2,485	31,332	543
13	13,565	7,499	3,213	64,734	1,563
14	5,183	1,880	783	23,604	513
15	5,895	4,225	4,557	33,081	539

scale efficiencies of this airline? What is its most productive scale size?

Which airlines are efficient according to this data set? For each, prepare a table showing the inefficient airlines for which the efficient airline is a peer.

8. The data, choice of inputs and choice of output in this exercise are based on the article, Shafer and Byrd (2000). The efficiency of organizational investments in information technology firms need to be studied. The three inputs related to investments in IT exercise are: information system (IS) budget as a percentage of sales, an organization's total processor value as a percentage of sales, and the percentage of the IS budget allocated to training. Because of a wide range of companies (in terms of sales, profits, number of employees, and industry) all three inputs were normalized to facilitate comparisons. Specifically, IS budget and an organization's total processor value were taken as a percentage of sales to normalize the differences in company size. Similarly, training expenditures were taken as a percentage of the total IS budget. Normalized measures are used to represent two outputs. Data on inputs and outputs for 36 of the 209 companies considered in the article are shown in the table below.

A data set for information technology firms

Company Name	*IS Budget as Percentage of Revenues*	*Processer Value as Percentage of Revenues*	*Training Budget as Percentage of IS Budget*	*Five Year Compound Annual Revenue Growth*	*Five Year Compound Annual Income Growth*
Ahawmut Natl. Corp.	5.69	0.46	0.93	29.4	135.6
AMR Corp.	6.70	0.70	0.57	26.3	114.1
Arvin Industries Inc.	1.70	0.78	3.47	28.7	108.6
Bank of Boston Corp.	3.28	0.72	1.89	42.0	135.7
Bankers Trust New York	5.80	1.02	1.45	32.5	123.2
Barnett Banks Inc.	3.17	0.88	2.67	28.6	94.6
Chase Manhattan Corp.	3.92	0.65	1.67	25.9	118.8
Chrysler Corp.	0.55	0.32	5.00	32.8	149.5
Comerica Inc.	2.61	1.22	2.67	41.5	124.7
Continental Bank Corp.	2.48	0.51	1.45	11.8	95.0
Corestates Financial Corp.	4.11	1.07	1.17	32.3	103.9
Dana Corp.	2.24	0.51	3.10	25.9	91.0
Delta Air Lines Inc.	2.01	0.73	2.00	25.9	105.5
E-Systems Inc.	0.82	1.12	3.67	38.8	131.5
Eaton Corp.	2.60	0.62	2.50	27.4	86.3
First of America Bank Corp.	2.75	0.82	2.47	34.1	99.1
First Union Corp.	3.36	0.48	0.97	42.6	115.1
Ford Motor Co.	1.66	0.51	4.00	30.2	103.4
Gencorp Inc.	2.15	0.91	3.80	25.5	33.2
General Dynamics Corp.	4.92	2.03	1.33	1.0	97.8
General Motors Corp.	2.57	0.84	3.45	29.1	100.7
KeyCorp	3.52	0.89	1.17	45.7	113.3
Lockheed Corp.	3.44	2.69	3.67	33.1	347.1
Martin Marietta Corp.	5.06	1.44	1.33	38.9	76.1

Company Name	*IS Budget as Percentage of Revenues*	*Processer Value as Percentage of Revenues*	*Training Budget as Percentage of IS Budget*	*Five Year Compound Annual Revenue Growth*	*Five Year Compound Annual Income Growth*
McDonnell Douglas Corp.	4.38	1.63	2.83	25.7	143.4
Mellon Bank Corp.	7.05	2.21	1.33	28.0	109.4
Meridian Bancorp Inc.	2.63	0.57	1.70	32.2	103.8
Paccar Inc.	2.00	1.11	3.33	27.2	100.9
Raytheon Co.	1.84	0.48	2.30	33.3	128.4
Republic New York Corp.	0.82	0.43	0.67	27.7	178.8
Sequa	0.98	0.77	2.33	28.7	89.9
Southwest Airlines Co.	2.29	0.59	2.33	43.7	118.1
Suntrust Banks	3.22	0.93	2.33	26.2	90.3
The Boeing Co.	5.30	2.84	1.85	34.1	93.6
UAL	2.96	0.42	1.19	25.0	111.9
US Air Group Inc.	1.57	0.39	1.93	18.7	82.7

Write all the eight DEA programs (Envelopment: input oriented and output oriented, Multiplier: output maximizing and input minimizing, and Returns to scale: constant and variable) for estimating the efficiency of Bank of Boston.

Solve these programs using an LP software. Tabulate the results. Is Bank of Boston efficient? If not, which company/companies would you recommend Bank of Boston consider emulating to improve the efficiency of its operation?

Which of these companies are efficient? For each of the efficient companies, prepare a table showing the inefficient companies for whom the efficient company is a peer.

9. In this exercise, DEA is applied to assess the performance of some banks. Three inputs (number of employees, capital, and deposits), and two outputs (loans and investments) are used for the purpose. Capital is measured by the book value of fixed assets and premises, and deposits are measured by the sum of long-term and saving deposits. Similarly, loans include loans to individuals, real estate loans, and commercial and industrial loans. Investments are measured using the value of all

Bank	Loans	Investments	Employees	Capital	Deposits
1	945.326	233.366	520	91.087	3,457.951
2	85.545	20.343	43	8.287	299.820
3	1,200.333	323.973	643	109.863	4,203.085
4	12.534	4.752	21	0.996	40.653
5	43.984	8.874	19	3.984	198.353
6	249.876	40.983	112	19.469	892.073
7	546.987	98.730	286	25.084	1,417.933
8	325.651	75.926	215	20.651	999.984
9	1,513.832	387.341	680	121.792	4,802.963

securities, other than those held in a bank's trading accounts. This data is shown below. Capital, deposits, loans and investments are measured in appropriate money units (US$).

Write all the eight DEA programs (Envelopment: input oriented and output oriented, Multiplier: output maximizing and input minimizing, and Returns to scale: constant and variable) for estimating the efficiency of Bank 2. Is this an efficient bank? If not, which bank/banks it should consider emulating to improve the efficiency of its operation?

Which are the efficient banks? For each of the efficient banks, prepare a table showing the inefficient banks for which the efficient bank is a peer.

10. The management of the Sun Restaurant company wants to analyze the efficiency of the operations of its seven fast-food restaurants. For the study of efficiency, the management has chosen the following inputs and outputs: number of employees (input), monthly expenses (input), monthly profit (output), and market share in percentage (output). Data for the input and output measures is shown below. Expenses and profits are measured in appropriate money units (US$).

Branch	Number of Employees	Monthly Expenses	Monthly Profit	Market Share
1	25	2,000	7,500	45
2	200	18,000	79,500	53
3	45	3,000	8,400	25
4	10	800	4,600	10
5	29	2,300	6,000	5
6	12	1,000	3,000	45
7	89	8,000	40,550	67

Write all the eight DEA programs (Envelopment: input oriented and output oriented, Multiplier: output maximizing and input minimizing, and Returns to scale: constant and variable) for estimating efficiency of Branch 5.

Solve these programs using an LP software. Tabulate the results. Is Branch 5 efficient? If not, which company/companies would you recommend the branch consider emulating to improve the efficiency of its operation?

Which are the efficient branches in this data set? For each of the efficient branches, prepare a table showing the inefficient branches for which the efficient branch is a peer.

4

Miscellaneous DEA Models and Recent Developments

The DEA model studied so far can be termed as the basic DEA models. Several variations of the basic models have been proposed in the DEA literature over the years. A brief outline of some of these models and recent developments is provided in this chapter. Those interested in knowing more about the models discussed here should refer to the references provided at appropriate places.

4.1 Multiplicative DEA Models

In the DEA models we have seen so far, the inputs and outputs of a DMU are aggregated additively. An alternative method of multiplicative aggregation is possible. Using this mode of aggregation, multiplicative DEA models have been constructed. The method of multiplicative aggregation makes sense, especially for manufacturing firms, because of the domination of the Cobb-Douglas production functions.

The Cobb-Douglas function is a popular form for production functions. With arguments $X = (X_1, \ldots, X_n)$, the function is expressed as

$$F(X) = A\prod_i X_i^{\alpha_i} \tag{4.1}$$

where $S_i a_i = 1$ and A is a positive constant. This function represents a typical example of multiplicative aggregation, with weights forming the indices.

In line with Equation 4.1, a multiplicative aggregation of virtual inputs can be defined as $\prod_i X_i^{u_i}$. (Contrast it with the additive aggregation, $S_i u_i X_i$.) When we use multiplicative aggregation, the efficiency of Firm A can be written as follows.

$$\boldsymbol{E}_A = \frac{1.8^{v_{VA,A}}}{8.6^{u_{CAP,A}} \times 1.8^{u_{EMP,A}}} \tag{4.2}$$

Note that the multiplicative aggregation can be easily transformed to additive one by logarithmic transformation.

$$\log E_A = v_{VA,A} \log 1.8 - (u_{CAP,A} \log 8.6 + u_{EMP,A} \log 1.8) \tag{4.3}$$

Note that this does not result in a fractional program, and hence can be directly used as a DEA objective function. This program has to be maximized subject to the condition that the efficiencies of all the firms are less than or equal to one.

The complete multiplicative DEA model is the following.

$$\begin{aligned}
&\max v_{VA,A} \log 1.8 - \left(u_{CAP,A} \log 8.6 + u_{EMP,A} \log 1.8\right) \\
&\text{subject to} \\
&v_{VA,A} \log 1.8 - \left(u_{CAP,A} \log 8.6 + u_{EMP,A} \log 1.8\right) \le 0 \\
&v_{VA,A} \log 0.2 - \left(u_{CAP,A} \log 2.2 + u_{EMP,A} \log 1.7\right) \le 0 \\
&v_{VA,A} \log 2.8 - \left(u_{CAP,A} \log 15.6 + u_{EMP,A} \log 2.6\right) \le 0 \\
&v_{VA,A} \log 4.1 - \left(u_{CAP,A} \log 31.6 + u_{EMP,A} \log 12.3\right) \le 0 \\
&v_{VA,A}, u_{CAP,A}, u_{EMP,A} \ge 1
\end{aligned} \tag{4.4}$$

Note the new requirement that the decision variables (i.e., weights) are constrained to be greater than unity. This is a requirement of the original model developed in Charnes et al. (1982; 1983). Note that, otherwise, we cannot write its dual! Note also that the dual will not include the variable q, but will incluide only the slack variables of the dual constraints.

For the sake of completeness, the dual form of the multiplicative DEA model is given here.

$$
\begin{aligned}
&\min -t_{VA,A} - s_{CAP,A} - s_{EMP,A}\\
&\text{such that}\\
&\log 1.8\lambda_{AA} + \log 0.2\lambda_{BA} + \log 2.8\lambda_{CA} + \log 4.1\lambda_{DA} - t_{VA,A} = \log 1.8\\
&\log 8.6\lambda_{AA} - \log 2.2\lambda_{BA} - \log 15.6\lambda_{CA} - \log 31.6\lambda_{DA} + s_{CAP,A} = \log 8.6\\
&\log 1.8\lambda_{AA} - \log 1.7\lambda_{BA} - \log 2.6\lambda_{CA} - \log 12.3\lambda_{DA} + s_{EMP,A} = \log 1.8\\
&\lambda_{AA}, \lambda_{BA}, \lambda_{CA}, \lambda_{DA} \geq 0\\
&t_{VA,A}, s_{CAP,A}, s_{EMP,A} \geq 0
\end{aligned}
\tag{4.5}
$$

4.2 Additive Models

Note that there is a slight difference in the formulation of the multiplicative models from the models discussed in Chapters 2 and 3. Unlike those models, the multiplicative models are not formulated fractional programs, and hence no normalization constraint is needed. Similar to the multiplicative model, it is easy to write an additive DEA model also. The additive version was published in Charnes et al. (1985b).

The general BCC version of the additive multiplier and envelopment versions can be written as follows.

Multiplier version

$$
\begin{aligned}
&\max_{u,v,v_0} z = V_m^T Y_m - U_m^T X_m + v_{0m}\\
&\text{such that}\\
&V_m^T Y - U_m^T X + v_{0m} \leq 0\\
&V_m^T, U_m^T \geq 1; \quad v_{0m} \text{ free}
\end{aligned}
\tag{4.6}
$$

Envelopment version

$$
\begin{aligned}
&\min_{\lambda,s,t} -t-s \\
&\text{such that} \\
&Y\lambda - t = Y_m \\
&X\lambda + s = X_m \\
&e^T\lambda = 1 \\
&\lambda, t, s \geq 0
\end{aligned}
\tag{4.7}
$$

4.3 Time Series Analysis using DEA

So far, we have compared the performance of a number of DMUs that operate at a particular point in time. This kind of analysis is normally referred to as a cross-sectional analysis. In contrast, one can think of comparing performance of DMUs over time. This type of analysis is generally referred to as a *time series analysis*.

In practice, DMUs are observed over multiple time periods; the variations of efficiency of DMUs over time can help in making important conclusions.

There are at least two ways of using DEA in a time series mode: *Window Analysis* and *Malmquist Productivity Index.*

4.3.1 Window Analysis

This mode is a moving average pattern of analysis, and is described in Charnes et al. (1985a). A DMU in each period is treated as if it is a different DMU. The performance of a DMU is compared with its performance in other periods, in addition to comparing it with the performance of other DMUs in the same period.

Let us consider the performance of four banks, A, B, C, and D, over a seven-year time period. Then, we select a three-year 'Window' (the analog of 'moving average' in traditional time series econometric analysis).

We analyze the firms for the first three years. In total, we will have 4 ´ 3 = 12 DMUs since Bank A in Year 1 is treated as a different DMU as compared to Bank A in Year 2. The DEA results may be presented in Table 4.1.

Table 4.1 DEA efficiencies of four banks for years 1, 2 and 3 (hypothetical data)

Bank	*Year 1*	*Year 2*	*Year 3*
A	0.981	0.973	0.981
B	1.000	0.997	0.991
C	0.937	0.913	0.952
D	1.000	1.000	0.998

Thus Bank B in Year 1, Bank D in Years 1 and 2 are the most efficient ones in this three-year window.

Then the window is shifted by one year, and DEA analysis is performed for the four banks for the Years 2, 3 and 4. We again have 12 DMUs. Their efficiencies are shown in Table 4.2.

Table 4.2 DEA efficiencies of four banks for years 2, 3 and 4 (hypothetical data)

Bank	*Year 2*	*Year 3*	*Year 4*
A	0.967	0.964	0.946
B	1.000	0.991	1.000
C	0.918	0.956	0.943
D	1.000	0.991	1.000

Similar analysis is performed for other windows, and the results are arranged in Table 4.3.

This window-type presentation facilitates an easy comparison of the performance of a bank over time as well as a comparison with its competitors at a particular point in time.

4.3.2 Malmquist Productivity Index Approach

Another method of time series analysis in DEA is to use the results of DEA in conjunction with the *Malmquist Productivity Index* (MPI) (see Fare et al. 1995). Note that this index number approach of analyzing performance variations over time was in existence much before DEA was proposed.

The output based Malmquist Productivity Index is defined as follows (Malmquist 1953).

$$M^{t+1}(x^{t+1}, y^{t+1}, x^t, y^t) = \left[\frac{D^t(x^{t+1}, y^{t+1})}{D^t(x^t, y^t)} \times \frac{D^{t+1}(x^{t+1}, y^{t+1})}{D^{t+1}(x^t, y^t)} \right]^{½} \tag{4.8}$$

Table 4.3 DEA efficiencies of four banks for years 1–7 using a three-year window (hypothetical data)

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*
A	0.981	0.973	0.981				
		0.967	0.964	0.946			
			0.991	0.973	0.981		
				0.967	0.888	1.000	
					0.756	1.000	0.993
B	1.000	0.997	0.991				
		1.000	0.991	1.000			
			0.948	0.796	0.848		
				0.856	0.888	0.728	
					0.869	0.809	0.967
C	0.937	0.913	0.952				
		0.918	0.956	0.943			
			1.000	0.867	0.800		
				0.856	0.743	1.000	
					0.856	0.938	0.725
D	1.000	1.000	0.998				
		1.000	0.991	1.000			
			0.874	0.767	1.000		
				0.793	0.896	0.847	
					0.971	0.744	1.000

where D^t is a *distance function* measuring the efficiency of conversion of inputs x^t to outputs y^t during the period t.

Note that if there is a technological change during the period $(t + 1)$, then,

$$D^{t+1}(x^t, y^t) = \text{Efficiency of conversion of input at period } t \text{ to output at period } t$$
$$1\ D^t(x^t, y^t) \qquad (4.9)$$

MPI is a geometric average of the effect of technology change. It can be written as,

$$M^{t+1}(x^{t+1}, y^{t+1}, x^t, y^t) = \frac{D^{t+1}(x^{t+1}, y^{t+1})}{D^t(x^t, y^t)}\left[\frac{D^t(x^{t+1}, y^{t+1})}{D^{t+1}(x^{t+1}, y^{t+1})} \times \frac{D^t(x^t, y^t)}{D^{t+1}(x^t, y^t)}\right]^{1/2}$$

or

$$M = E\ ´\ T \qquad (4.10)$$

where E = Technical efficiency change, and
T = Technology change.

Consider Figure 4.1. S^{t+1} is the frontier at $(t + 1)$. If there is a technical progress, S^{t+1} will shift upwards from S^t. M represents achievement (x^{t+1}, y^{t+1}) at time $(t + 1)$, while N represents the achievement at time t.

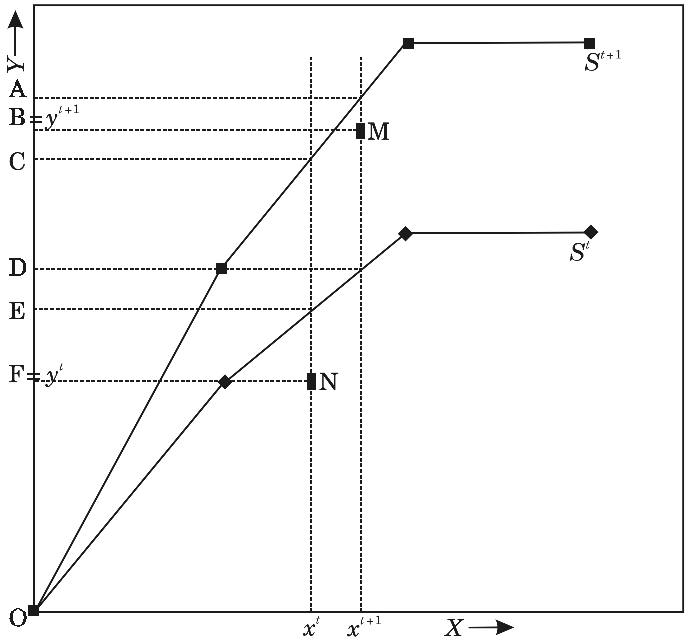


Figure 4.1 Malmquist productivity index

Note that DEA efficiency can be considered as a distance measure because it reflects the efficiency of conversion of inputs to outputs. Hence,

$$D^{t+1}(x^{t+1}, y^{t+1}) = \text{DEA efficiency using } x^{t+1} \text{ inputs and } y^{t+1} \text{ outputs}$$

$$= \frac{OB}{OA}$$

Similarly,

$$D^t(x^t, y^t) = \frac{OF}{OE}$$

Hence,

$$\begin{aligned} E &= \text{Technical efficiency change} \\ &= \frac{D^{t+1}(x^{t+1}, y^{t+1})}{D^t(x^t, y^t)} \\ &= \frac{(OB/OA)}{(OF/OE)} \end{aligned} \tag{4.11}$$

If $E > 1$, then there is an increase in the technical efficiency of converting inputs to outputs.

What does $[D^t(x^t, y^t)]/[D^{t+1}(x^t, y^t)]$ in Equation 4.10 mean? Because of a technical change, the same input x^t can produce a higher output when used during the period $(t + 1)$. Input x^t can produce only OE as its best ouput in time t, but can produce a higher output OC in time $(t + 1)$. Hence, the ratio (OC/OE) represents a measure of technology change. If this ratio is greater than unity, then there is proof of technological improvement.

$$D^t(x^t, y^t) = \frac{OF}{OE}$$

$$D^{t+1}(x^t, y^t) = \frac{OF}{OC}$$

Hence,

$$\frac{D^t(x^t, y^t)}{D^{t+1}(x^t, y^t)} = \frac{OC}{OE}$$

For technological progress, this ratio should be greater than unity. Similarly,

$$\frac{D^t(x^{t+1}, y^{t+1})}{D^{t+1}(x^{t+1}, y^{t+1})} = \frac{OA}{OD} > 1$$

for technological progress.

Thus,

$$
\begin{aligned}
T &= \text{Technology change} \\
&= \left[\frac{D^{t}\left(x^{t+1}, y^{t+1}\right)}{D^{t+1}\left(x^{t+1}, y^{t+1}\right)} \times \frac{D^{t}\left(x^{t}, y^{t}\right)}{D^{t+1}\left(x^{t}, y^{t}\right)}\right]^{1/2} \\
&= \left[\frac{OA}{OD} \times \frac{OC}{OE}\right]^{1/2}
\end{aligned}
\tag{4.12}
$$

represents the average technological change, measured as the geometric mean of the foregoing two ratios.

4.4 Some Extensions of DEA

When the basic DEA models are applied in practical problems, several issues are encountered. Therefore, the DEA models often have to be modified to extend their applicability to these problems. Some of these extensions reported in the literature are discussed here.

4.4.1 Non-discretionary Inputs and Outputs

The basic DEA models we have discussed so far assume that all inputs and outputs are discretionary, i.e., all of them are under the control of the management of the DMU, and can be varied at its discretion. Thus the failure of a DMU to produce maximal output levels with minimal input consumption results in decreased efficiency score. However, in practice, there may be circumstances when some of the inputs or outputs are beyond the control of the management of a DMU. For example, a school situated in a backward area will have a greater enrolment of socially weaker students. A manager of agricultural farms cannot alter soil characteristics.

There are at least two ways to approach this issue in DEA.

The first method introduces non-discretionary characteristics within the DEA framework. It is done by recognizing that there cannot be a best achievable performance for non-discretionary inputs or outputs. In the DEA framework, the envelopment version has the flexibility to handle this situation.

Let I_D be the set of discretionary inputs, where I_D Î I and let $I_{ND} = I - I_D$. The input oriented envelopment problem now becomes,

$$
\begin{aligned}
& \min \theta_m \\
& \text{such that} \\
& \sum_{n=1}^{N} y_{jn}\lambda_n \geq y_{jm}; \quad j = 1, 2, \mathrm{K}, J \\
& \sum_{n=1}^{N} x_{in}\lambda_n \leq \theta_m x_{im}; \quad i \in I_D \ (\leftarrow \text{only for discretionary inputs}) \\
& \sum_{n=1}^{N} x_{in}\lambda_n \leq x_{im}; \quad i \notin I_D (\leftarrow \text{for non-discretionary inputs}) \\
& \lambda_n \geq 0 \\
& \theta_m \text{ free}
\end{aligned}
\qquad (4.13)
$$

Obviously, the above model will consider only non-discretionary inputs. To consider non-discretionary outputs, one has to use the output oriented envelopment version.

There is another way of handling non-discretionary inputs, which has been used by many DEA researchers. This method draws from published literature on *Total Factor Productivity* (Caves et al. 1982). It combines DEA and regression. Non-discretionary variables are handled outside the DEA framework.

In this method, the efficiencies of DMUs are obtained without considering the non-discretionary variables. Then, the resulting DEA efficiencies are regressed using the non-discretionary variables as the independent variables. This process filters the effects of non-discretionary variables on the efficiency ratings. If needed, the residuals of regression may be used as the final efficiency scores of the DMUs. An example of this approach is described in Section 6.2 in the context of the application of DEA to transport undertakings and schools.

4.4.2 Categorical Inputs and Outputs

We have so far considered all inputs and outputs to be continuous variables. However, practical applications do involve categorical or ordinal variables.

As an example, some DMUs can perform well simply by virtue of their location. If the locations could be characterized as good, medium or poor, it may not be correct to compare the performance of DMUs located in a good location with those located in a poor location. Decision-making units having similar location characteristics should be compared, and caution must be exercised in comparing DMUs that have different location characteristics.

Similarly, some DMUs may have a particular capability unlike others. For example, some hotels may have a drive-in facility. It may not be meaningful to compare the performance of these DMUs with those who do not have this facility.

This issue is tackled by comparing only those DMUs having a particular capability. Suppose that only a subset N_A of the DMUs has a particular capability, where N_A Î N. Then the envelopment DEA program can be modified as follows (Banker and Morey 1986; Charnes et al. 1994).

$$
\begin{aligned}
&\min \theta_m \\
&\text{such that} \\
&\sum_{n \in N_A} y_{jn} \lambda_n \geq y_{jm}; \quad j = 1, 2, \mathrm{K}, J \\
&\sum_{n \in N_A} x_{in} \lambda_n \leq \theta_m x_{im}; \quad i = 1, 2, \mathrm{K}, I \\
&\lambda_n \geq 0 \\
&\theta_m \text{ free}
\end{aligned}
\tag{4.14}
$$

This DEA model compares only those DMUs that possess similar characteristics. Hence, the resulting efficiency scores provide a more accurate measurement of performance.

4.4.3 Incorporating Judgements and *a Priori* Knowledge

A DEA program is a mathematical program, which yields efficiency ratings when supplied with input–output data. However, DEA can provide results that are unacceptable on the basis of some other criteria or inputs. Some specific situations are:

(*a*) The DEA analysis ignores information that cannot be directly incorporated into the model or that contradicts expert opinion.
(*b*) The management of a DMU have some *a priori* information regarding the weights to be given to different inputs and outputs. More generally, they may have a range for the weights.
(*c*) DEA may sometimes fail to discriminate among DMUs, especially for a small sample size. Hence, all may be rated efficient (e.g., Firms A, B, C and D under VRS assumption in Chapter 1).

These requirements are normally addressed by imposing additional constraints in the DEA program. These constraints may restrict the weights of inputs and outputs, weights of DMUs, or impose other constraints that are meaningful for the problem.

For instance, from a set of hundreds of DMUs, one may like to compare only those which consume (or produce) more than a certain quantity of input (or output) for a specific situation. This can be done simply by introducing another constraint in the model. As an example, Dyson and Thanassoulis (1988) have imposed upper and lower bounds on the weights.

The method of 'assurance region' (Thompson et al. 1986) has been developed by restricting the relative magnitude of the weights for special items. For example, the ratio of weights for the two inputs (capital employed and number of employees) may be restricted by providing this additional constraint to the output maximizing multiplier version:

$$lowerbound \text{ £ } u_{CAP,\ A}/u_{EMP,\ A} \text{ £ } upperbound$$

(here Firm A is the reference firm).

The method of 'cone-ratio envelopment' described as cone-ratio DEA model in Charnes et al. (1989) and polyhedral cone-ratio DEA model in Charnes et al. (1990) uses more general framework to specify weight restrictions. Weight restrictions are incorporated using the form $\{Cw$ ³ $0\}$ where w is a column matrix of the weights ($u_{CAP,\ A}$, $u_{EMP,\ A}$, etc.) and C is a matrix where weight restrictions are specified. It is possible to write the weight restriction constraints of the assurance region in the form $\{Cw\ ^{3}\ 0\}$. Thus, the cone-ratio envelopment method is more general compared to the assurance region method. More complicated cone-ratio formulations are also available in the literature. It has been proved that the cone-ratio DEA model can be transformed to a form similar to the CCR DEA model using suitable transformations, and hence can be solved with the existing DEA software without any need for modifying computer codes. Interested readers should refer to Charnes et al. (1989; 1990) and Brockett et al. (1997).

4.5 Exercises

1. The following table gives two outputs and two inputs used for measuring the relative efficiency of some highway maintenance patrols. All the patrols are approximately similar in size. Each patrol is responsible for a fixed number of lane-kilometres of highway, as well as the activities associated with that portion of the network.

   (*a*) Identify efficient patrols, and the most productive scale size for each of the inefficient patrols.

   (*b*) The management of the DMUs feels that the weight of the available Average Traffic Served should not be less than half, but cannot be more than 20 per cent, of the weight assigned to the variable Area Served Factor. Estimate the relative efficiency of the patrols after imposing these restrictions.

	Outputs		*Inputs*	
	Area Served Factor	*Average Traffic Served*	*Maintenance Expenditure*	*Capital Expenditure*
PATROL 1	0.5	8.0	930	600
PATROL 2	3.5	62.0	256	600
PATROL 3	1.0	75.0	990	600
PATROL 4	2.3	62.0	76	2,000
PATROL 5	4.5	48.0	20	700
PATROL 6	0.8	40.0	20	4,000
PATROL 7	3.4	3.3	35	900
PATROL 8	0.8	12.0	10	2,400

2. The following table summarizes the performance of selected State Transport Undertakings (STUs) in India. Fleet size is a representative of the capital input, total number of staff is a representative of the labour input, and diesel consumption represents a dominant material input. Passenger kilometres, a composite measure of the passenger carried and the average distance travelled by passengers, is considered as the single measure of output.[1]

Unit	*Fleet Size (no.)*	*Total Staff (no.)*	*Diesel Consumption (kilolitres)*	*Passenger Kilometres (millions)*
STU1	15,483	1,11,979	3,17,679	53,409
STU2	15,235	1,19,630	3,28,010	64,486
STU3	9,899	63,712	2,16,159	37,460
STU4	8,945	59,706	1,86,557	29,485
STU5	8,023	56,864	1,44,478	24,053
STU6	4,115	25,892	90,268	15,903
STU7	2,368	12,551	54,411	7,633
STU8	1,290	8,338	42,117	8,147
STU9	1,150	8,992	34,971	6,939
STU10	940	8,733	49,813	7,559
STU11	947	6,707	19,637	2,500
STU12	954	7,223	18,244	2,435
STU13	809	6,078	26,776	5,596
STU14	766	6,069	25,134	4,804
STU15	700	5,092	23,372	5,062
STU16	700	5,268	23,394	4,819
STU17	661	4,902	20,220	4,411
STU18	644	4,668	21,545	4,426
STU19	593	4,470	18,326	3,603
STU20	584	4,116	19,163	4,394
STU21	1,598	9,179	35,830	3,775
STU22	198	1,045	1,432	143
STU23	198	1,114	1,030	57
STU24	130	802	647	73
STU25	3,706	34,220	69,710	11,722
STU26	3,067	38,965	81,009	11,060
STU27	2,094	19,244	45,225	10,131
STU28	753	6,105	11,815	1,795
STU29	117	1,291	2,652	417

[1] This application is partly described in Section 6.2.

Perform data envelopment analysis to identify efficient and inefficient STUs.

Note that STUs 21 to 24 operate on a hilly terrain, while STUs 25 to 29 operate exclusively in urban areas. The remaining operate mostly long distance, inter-city bus services. There is reason to believe that STUs operating on a hilly terrain are at a disadvantage compared to others. Also, STUs operating in urban areas have a special advantage over others as they have better capacity utilization. How can you incorporate these differences into the DEA analysis?

STUs 1, 4, 10, 16, 24, and 25 are Government-owned, while the remaining STUs operated independently as private corporations. How will you test the hypothesis that private corporations generally perform better than Government departments?

3. The following table provides data regarding physical performance and energy consumption in road and rail transport. PKM refers to Passenger Kilometres, a composite measure of the number of passengers and the distance travelled by passengers, while TKM refers to Tonne-Kilometres, which is a measure of total tonnage of materials and the distance transported. These are the output measures, while energy consumption is a dominant material input.[2]

	Railways			*Roadways*		
Year	*Energy (TJ)*	*PKM (billion)*	*TKM (billion)*	*Energy (TJ)*	*PKM (billion)*	*TKM (billion)*
1	2,76,547	209	159	4,42,590	353	98
2	2,88,503	221	174	4,59,360	377	103
3	2,84,951	227	178	4,98,588	408	106
4	2,75,747	223	178	5,30,511	448	116
5	2,59,278	227	182	5,73,339	486	124
6	2,54,974	241	206	6,27,644	850	193
7	2,33,496	257	223	6,83,686	893	210
8	2,23,517	269	231	7,55,410	980	238
9	2,07,529	264	230	8,29,482	905	275
10	1,25,367	296	257	11,45,102	1,500	350

[2] This application is partly described in Section 6.2.

The DEA efficiencies provide an idea of the energy efficiency of the two modes of transport over time. Perform window analysis with a three-year window, and make useful conclusions.

4.6 Other DEA Models and Extensions[3]

The literature on DEA is continuously evolving and contains many more extensions, including the free disposal hull (FDH) (Tulkens 1993), generalized DEA (Nakayama et al. 2000), value efficiency analysis (Halme et al. 1999), etc. The FDH model is quite similar to the BCC model, except that the multipliers (l) are restricted to binary (zero and one) variables.

Research studies on the incorporation of preference information in DEA (e.g., the value efficiency analysis suggested in Halme et al. 1999) have largely explored the relationship between DEA and the larger field of *Multi-Criteria Decision-Making* (MCDM).[4] Though DEA was originally developed as a tool for performance measurement, over the last few years the linkages between the fields of DEA and MCDM have been explored (Stewart 1996; Agrell and Tind 1998; Belton and Stewart 1999). Data envelopment analysis is now widely accepted as a tool for multi-criteria decision-making. Stewart (1996) has compared and contrasted the traditional goals of DEA and MCDM. The goal of DEA is to determine the productive efficiency of a system or DMU by comparing how well the DMUs convert inputs into outputs, while the goal of MCDM is to rank and select from a set of alternatives that have conflicting criteria. Incorporating preference information in a DEA analysis extends it naturally towards MCDM. This issue has been discussed for a long time in DEA literature (Golany 1988; Thanassoulis and Dyson 1992; Zhu 1996; Halme et al. 1999). Golany (1988) and Thanassoulis and Dyson (1992), among others, have developed target setting models. Golany

[3] This section is meant for advanced readers, and can be skipped without any loss of continuity.

[4] A *multi-criteria decision-making* problem is characterized by the need to choose one or a few criteria from among a number of alternatives, and the choice is made on the basis of two or more criteria or attributes (Dyer et al. 1992).

(1988) has allowed the decision maker (DM) to select the preferred set of output levels, given the input levels of a DMU. Thanassoulis and Dyson (1992) have estimated alternative input/output target levels to render relatively inefficient DMUs efficient. Another approach to incorporate preference information in DEA is to restrict the flexibility of weights (e.g., Charnes et al. 1989). Generally speaking, weight restrictions result in the reduction of the number of efficient DMUs. Zhu (1996) proposed a model that calculates efficiency scores incorporating data regarding the DM's preference. In the value efficiency analysis proposed in Halme et al. (1999), the DMs' preferences are incorporated in DEA by explicitly locating his/her most preferred input–output vector on the efficient frontier. This results in the DM's *Most Preferred Solution*. The approach uses a pseudo-concave value function to estimate a hyperplane for comparing the utility of observations.

The field of DEA is developing very rapidly with new topics being added regularly. While most of the important issues are covered in the previous sections of this chapter and in the previous chapters, other topics will not be covered in this book. Interested readers are advised to refer to the articles cited in this chapter and elsewhere in this book. More bibliographic information on DEA is provided in Chapter 6.

5

Computer-based Support for DEA

In this computer era, it is important that any management science technique has adequate software support so that potential users are encouraged to use it. Software harnesses the computing power of PCs for use in practical decision-making situations. It can also expedite the implementation of a method. Potential users can access the software more easily if it is accessible from Web site on the Internet, and, if it is available as a freeware, downloaded.

5.1 Computational Features of DEA

Data envelopment analysis is based on linear programming. Therefore, any software package available for solving linear programming can in principle be used for solving DEA applications. However, there are some significant characteristics of DEA that cannot be handled satisfactorily by standard LP packages, thus necessitating specialized software. These characteristics include the following.

(*a*) DEA applications typically involve solving separate linear programming problems for each of the DMUs involved. If there are N DMUs, one has to invoke the LP package N times, each time modifying the objective function and other parameters. This is tedious and time-consuming.
(*b*) Multiplier DEA versions involve normalizing constraints. For example, the output maximizing multiplier problem

requires the following constraint that normalizes the weighted sum of inputs.

$$U_m^T X_m = 1$$

Obviously, the weights u_i are inversely proportional to the magnitude of inputs: if X_i is large, then the corresponding u_i have to be necessarily small. It is possible that if the value of inputs is sufficiently large, the resulting small values of weights can confound the testing of optimality in the LP algorithm.

(*c*) The optimal solution of a DEA model often involves many more zero values than encountered in regular linear programming models. For example, when an envelopment DEA model is solved for an efficient DMU, only the variable q and its corresponding l are positive, while all other values are equal to zero in the optimal solution. Given this situation, readers familiar with linear programming will realize that DEA models can lead to degeneracy. In linear programming, a basis corresponding to a feasible solution is said to be degenerate when at least one of the basic variables has a value of zero. When a DEA model is solved using an ordinary LP package, the basic solutions of many iterations can contain many zero basic variables, and therefore will be degenerate basic solutions. The main theoretical implication of degeneracy is the phenomenon of cycling—the simplex procedure would repeat the same sequence of iterations, never improving the objective value and never terminating the computations. Thus it is necessary to incorporate methods for eliminating cycling, resulting in the reduction of computational speed.

(*d*) As noted in Chapter 2, the use of non-Archimedean infinitesimals can create computational difficulties when solved using an LP software. Though the non-Archimedean infinitesimals are not numbers, standard LP packages do require that the infinitesimals be represented in the form of small numbers. The numeric values for non-Archimedean infinitesimals in actual computations should be chosen to be much smaller than other input and output values so that optimization is not affected. However, it is normally not possible to know the values of inputs and outputs in commercial software that will be used for a variety of applications. Therefore, solution methodologies that do not require explicit specification of the values of infinitesimals are required for use with DEA software. A two-phase optimization procedure, discussed in Chapter 2, can be used to avoid assigning numerical values to non-Archimedean infinitesimals (see Ali and Seiford 1993).

More information about the computational aspects of DEA is available in Ali (1994).

5.2 DEA Software

The DEA methodology can be programmed using various high-level programming languages to generate their software. Software implementation for DEA is available from several developers including universities and private companies. Examples include the University of Warwick and Banxia Software Ltd., UK. Similarly, the latter has developed *Frontier Analyst*. Demonstration versions of the above software are available from their respective developers. Other software for DEA include BYU-DEA developed at Brigham Young University, Utah, USA; IDEAS developed by 1 Consulting, Massachusetts, USA; and, PIONEER developed at the Southern Methodist University, Dallas, USA. More information on many of these software is available in Charnes et al. (1994). Other free DEA software include Efficiency Measurement System (EMS) developed at the University of Dortmund, Germany, and A Data Envelopment Analysis (Computer) Program (DEAP) developed at the Centre for Efficiency and Productivity Analysis, University of New England, Australia. Note that the DEA software listed above are available free only for academic purposes.

The latest survey of decision analysis software (Maxwell 2000) published in *ORMS Today* includes a description of software such as Frontier Analyst.

5.3 Internet Support for DEA[1]

As mentioned earlier, Internet support for software including DEA software will augment their capabilities. Internet support provides avenues for faster communication between users and developers, incorporation of feedback from users, and quick updating of new versions of DEA software.

Developers of most of the DEA software mentioned in Section 5.2 provide elaborate Internet support to their clients through Web sites. Support includes providing information, and making demonstration versions of the software available. The following are the Internet sites for some of the DEA software.

(*a*) Frontier Analyst software, Banxia Software Ltd., Glasgow, Scotland (*http://www.banxia.com*).
(*b*) Warwick-DEA software, University of Warwick, UK (*http:// www.warwick.ac.uk/~bsrlu/*).
(*c*) The EMS software, University of Dortmund, Germany (*http://www.wiso.uni-dortmund.de/lsfg/or/scheel/ems/*).
(*d*) The DEAP software, Centre for Efficiency and Productivity Analysis, University of New England, Australia (*http://www.uq.edu.au/~uqtcoell/deap.htm*).
(*e*) The DEA-Solver-PRO software developed by SAITECH, Inc., New Jersey, USA (*http://www.saitech-inc.com/security/dea.htm*). This software is bundled with a recent book on DEA (Cooper et al. 2000).
(*f*) OnFront software developed by Economic Measurement and Quality AB, Box 2134, S-220 02 Lund, Sweden (*http://www.emq.se/software.html*).
(*g*) The DEAFrontier software, bundled with Zhu (2002), is an Add-In for Microsoft Excel for performing DEA computations (*http://www.deafrontier.com*).

Charnes et al. (1994) provide the e-mail addresses of developers of some other DEA software. In addition, the softwares surveyed at the *ORMS Today* are available at the Internet site: *http://www.lionhrtpub.com/software-surveys.shtml*. Herrero and Pascoe (2002) provide a

[1] I thank Mr Jukka Pattero of the Helsinki University of Technology for his help in compiling some of the Internet facilities for DEA.

comparative analysis of many different DEA software. Many of the DEA software are also listed at the Internet site *http://www.deazone.com/software/index.htm.*

Note that Internet support need not be limited to DEA software. Internet provides several other support facilities for DEA. They include the following features.

(*h*) A Web-based DEA interface is available at the University of Mannheim, Germany (*http://heliodor.bwl.uni-mannheim.de/webdea.html*). The interface enables users of DEA to enter data for their DEA computations online, processes their data immediately, and provides DEA efficiencies and other relevant results.

(*i*) A number of Internet sites provide DEA tutorials. Examples include, among others, *http://www.emp.pdx.edu/dea/homedea.html* (Portland State University, Oregon, USA), *http://www.warwick.ac.uk/~bsrlu/dea/deat/deat1.htm* (University of Warwick, UK), *http://www.ms. ic.ac.uk/jeb/or/dea.html* (Imperial College, London, UK), and, *http://members.tripod.com/moezh/DEAtutorial/DEAtutorial.html* (York University, Canada).

(*j*) Most of these Internet sites provide some bibliographical information on DEA and also links to other DEA sites. The Web site of the author of this book (*www.geocities.com/r_ramnath*) also provides information on DEA. A detailed DEA bibliography is available at the University of Warwick, UK (refer item *b*).

(*k*) Other interesting Internet sites for DEA include the Web site of the Productivity Analysis Research Network (PARN) in SDU-Odense University, Denmark (*http://www.busieco.ou.dk/parn/*), and the EURO Working Group on Data Envelopment Analysis and Performance Measurement (Euro-DEAPM) (*http://www.deazone.com/euro-deapm/*).

A collection of data sets for testing and practising DEA problems are available in many Internet sites. These include the DEA dataset repository (*http://www.etm.pdx.edu/dea/dataset/Default.htm*) maintained by the Portland State University, USA, and the University of Auckland, New Zealand, the OR-Library (*http://mscmga.ms.ic.ac.uk/jeb/orlib/deainfo.html*), and the data set maintained at the site *http://www.deazone.com/datasets/FILE1/index.asp*.

5.4 A Brief Description of Some DEA Software

In this section, a brief description of some free DEA software available on the Internet will be provided. The information has been obtained from the documentation available at the corresponding Internet sites. More details can be obtained from these sites. It is worth mentioning again that these software are available free only for academic purposes.

5.4.1 Efficiency Measurement System (EMS) Software

Efficiency Measurement System (EMS) (Version 1.3) is a software developed at the University of Dortmund, Germany, for Windows 9x/NT, for performing DEA computations. EMS is free of charge for academic users. It can be downloaded from: *http://www.wiso.uni-dortmund.de/lsfg/or/scheel/ems/*. The latest news about EMS, downloads and bug fixes can also be found at this page. Enquiries regarding the software can be sent to the developer at H.Scheel @wiso.uni-dortmund.de.

EMS uses LP Solver DLL BPMPD 2.11 by Csaba M′esz′aros for the computation of the scores (*http://www.netlib.org*). It is an interior point solver.

There is theoretically no limitation on the number of DMUs, inputs and outputs in EMS. The size of the analysis is limited by the memory of the computer in which the computations are performed. The developer of the software claims to have solved problems with over 5,000 DMUs and about 40 inputs and outputs successfully.

EMS accepts data in MS Excel 97 (and earlier versions) or in text format. Guidelines for using Excel files are listed below.

(*a*) The input–output data should be collected in one worksheet, 'Data'.
(*b*) Input names are specified by including the string '{I}' at the end of the names.

(*c*) Output names are specified by including the string '{O}' at the end of the names.
(*d*) The first row should contain names of inputs, followed by names of outputs.
(*e*) The first column of the worksheet should contain the DMU names.
(*f*) The worksheet should not contain formulae.

The following guidelines should be used for text files.

(*a*) If text files are used, the name of the text file should be mentioned in a separate file named *Schema.ini*. This file is normally available with the EMS software and may be modified whenever necessary.
(*b*) In a text file, columns are separated by tabs.
(*c*) The first column contains the DMU names.
(*d*) As with Excel files, the input names contain the string '{I}', while the output names contain the string '{O}'.

Non-discretionary inputs and outputs can also be specified in EMS. This is done by including the string '{IN}' instead of '{I}' for inputs and by including '{ON}' instead of '{O}' for outputs.

Weight restrictions (see Chapter 4) can be specified using a separate Excel or text file. The EMS software can also perform Window analysis and Malmquist index-based analysis. For more details, the reader is referred to the documentation available at the EMS home page specified earlier.

5.4.1.1 Solving a DEA example using EMS: Use of EMS for computing DEA efficiencies will be explained in this section using the example of the four DMUs A, B, C and D discussed in Chapter 1. Recall that there were the two inputs (CAP and EMP) and one output (VA) for the four firms.

Step 1—Data input: If a text file is used for inputting data, the following should be its contents. The symbol ® indicates a tab.

DMU ® CAP {I} ® VA {O}
A ® 8.6 ® 1.8 ® 1.8
B ® 2.2 ® 1.7 ® 0.2

C ® 15.6 ® 2.6 ® 2.8
D ® 31.6 ® 12.3 ® 4.1

Step 2—Making the calculations: EMS software can be started by clicking the appropriate icon. Input and output data can be loaded by pressing Ctrl+O or invoking the File menu, and then clicking ‘Load data’. Once the data are loaded, efficiencies can be calculated by pressing Ctrl+M or be invoking the DEA menu and the clicking ‘Run Model’.

Step 3—Viewing results: The output created by the EMS software when the input data is specified as the text file as in Step 1 is shown in Figure 5.1.

The window caption describes which model was computed. In this case, the window caption above is Data.txt_CRS_RAD_IN, which specifies that the input data file is Data.txt, and that the computations have been carried out assuming constant returns to scale, radial distance, and input orientation.

Step 4—Interpreting the results: The result table contains the following:

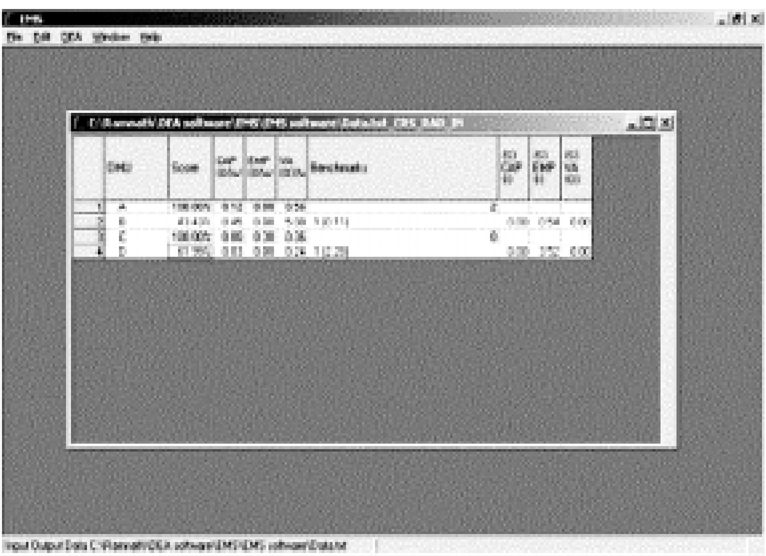


Figure 5.1 Output screen from EMS software

(*a*) DMU name

(*b*) DEA efficiency score. For example, the above output shows that the efficiency of DMU A and C are 100 per cent while those of DMUs B and D are 43.43 per cent and 62 per cent respectively. Please recall that these were the same efficiencies calculated in Chapter 1.

(*c*) Weights (shadow prices) {W} or virtual inputs/outputs {V} as selected in Menu DEA ® Format.

(*d*) Benchmarks:

(i) For inefficient DMU: the reference DMUs with corresponding intensities (the 'lambdas') in brackets.

(ii) For efficient DMU: the number of inefficient DMUs which have chosen the DMU as Benchmark.

(*e*) Slacks {S} or factors {F}. Depending of the chosen distance, for radial and additive measures, the slacks are displayed.

5.4.2 A Data Envelopment Analysis (Computer) Program (DEAP)

Another DEA software, A Data Envelopment Analysis (Computer) Program (DEAP) (Version 2.1) has been developed by Professor Tim Coelli of the Centre for Efficiency and Productivity Analysis, University of New England, Australia.[2]

Detailed instructions on how to use the DEAP software are available in a user guide (Coelli 1996), available when the software is downloaded. The documentation is a working paper (96/08) of the Centre for Efficiency and Productivity Analysis, Australia, and can also be downloaded separately.

Like the EMS software, theoretically there is no limitation on the number of DMUs, inputs and outputs in DEAP. The size of the DEA analysis is limited by the memory of computer on which the computations are performed. The DEAP software can consider a variety of DEA models. Three principal options are:

[2] As on May 2003, the Centre for Efficiency and Productivity Analysis has moved to the University of Queensland at *http://www.uq.edu.au/economics/cepa/deap.htm*. Professor Coelli can be contacted at *t.coelli@economics.uq.edu.au*.

(*a*) Standard CRS and VRS DEA models as discussed in Chapters 2 and 3.
(*b*) The extension of the above models to account for cost and allocative efficiencies. This information can be obtained from Fare et al. (1994).
(*c*) Application of Malmquist productivity index approach for time series analysis, as discussed in Chapter 4.

All methods are available as either input or output orientation models. The output from the program includes, where applicable, technical, scale, allocative and cost efficiency estimates; residual slacks; peers; and other indices.

DEAP is a DOS based computer program, but can be easily run from Windows operating systems. The program involves a simple batch file system where the user creates a data file (usually with the extension .DTA) and a small file containing instructions (with the extension .INS). The DEAP software reads the data, follows the instructions in the instruction file and produces the output of its analysis in an output file (with the extension .OUT). All these files (data, instruction and output) are simple text files, and must be stored in the same directory where the executable file of the software, DEAP.EXE, is stored.

In addition to the above files, the execution of DEAP Version 2 (on an IBM PC) involves the use of another file, DEAP.000. This file is supplied when the software is downloaded. According to the software's documentation (Coelli 1996), this file contains only the value of a variable used to test inequalities with zero.

Data must be listed by observation with one row for each firm. There must be a column for each output and each input, with all outputs listed first followed by inputs (from left to right across the file). As an example, let us consider the four DMUs A, B, C and D discussed in Chapter 1. Their DEA efficiencies need to be calculated, using a single output (VA) and two inputs (CAP and EMP). This involves the following steps.

Step 1: Create a data file listing the input and output values for the four firms. Its contents are shown below. Note that there are four rows, one each for DMUs A, B, C and D, and that output values are listed first followed by the two inputs.

1.8	8.6	1.8
0.2	2.2	1.7
2.8	15.6	2.6
4.1	31.6	12.3

The data file can be longer and more complicated depending on the nature of analysis. For example, the DEAP software can calculate cost efficiencies and Malmquist indices for time series analysis, and data must be entered in proper format. Please consult the software documentation for further information.

Step 2: Create an instruction file giving further details of the computation. The instruction file contains nine lines and provide necessary information on the names of data and output files, number of firms, number of time periods, number of inputs and outputs, input or output, orientation, CRS or VRS, etc. For example, the instruction file for calculating CRS efficiencies of the four DMUs should contain the following. The information is self explanatory, and the reader should consult the software documentation for further information.

DEAbook.dta	DATA FILE NAME
DEAbook.out	OUTPUT FILE NAME
4	NUMBER OF FIRMS
1	NUMBER OF TIME PERIODS
1	NUMBER OF OUTPUTS
2	NUMBER OF INPUTS
0	0 = INPUT AND 1 = OUTPUT ORIENTATED
0	0 = CRS AND 1 = VRS
0	0 = DEA(MULTI-STAGE), 1 = COST-DEA, 2 = MALMQUIST-DEA, 3 = DEA(1-STAGE), 4 = DEA(2-STAGE)

Step 3: Now calculate the DEA efficiencies. As mentioned earlier, DEA calculations can be performed by invoking the executable file DEAP.EXE. Once invoked, the software will prompt for the name of instruction file. The name of the instruction file should then be supplied on the screen (along with the .INS extension). DEAP will then take some time to make the DEA computations, and will create the output file (with the name as specified in the instruction file). The output file for the computation of DEA efficiencies of the four firms is shown below. The output is self explanatory.

Output file from DEAP
Results from DEAP Version 2.1 Instruction file = DEAbook.ins Data file = DEAbook.dta Input orientated DEA Scale assumption: CRS Slacks calculated using multi-stage method

EFFICIENCY SUMMARY:

firm	*te*
1	1.000
2	0.434
3	1.000
4	0.620
mean	0.764

SUMMARY OF OUTPUT SLACKS:

firm output:	1
1	0.000
2	0.000
3	0.000
4	0.000
mean	0.000

SUMMARY OF INPUT SLACKS:

firm input:	1	2
1	0.000	0.000
2	0.000	0.538
3	0.000	0.000
4	0.000	3.525
mean	0.000	1.016

SUMMARY OF PEERS:

firm	*peers:*
1	1
2	1
3	3
4	1

SUMMARY OF PEER WEIGHTS:
(in same order as above)

firm	*peer weights:*
1	1.000
2	0.111
3	1.000
4	2.278

PEER COUNT SUMMARY:
(i.e., no. times each firm is a peer for another)

firm	*peer count:*
1	2
2	0
3	0
4	0

SUMMARY OF OUTPUT TARGETS:

firm output:	1
1	1.800
2	0.200
3	2.800
4	4.100

SUMMARY OF INPUT TARGETS:

firm input:	1	2
1	8.600	1.800
2	0.956	0.200
3	15.600	2.600
4	19.589	4.100

FIRM BY FIRM RESULTS:
Results for firm: 1
Technical efficiency = 1.000

PROJECTION SUMMARY:

variable	*original value*	*radial movement*	*slack movement*	*projected value*
output 1	1.800	0.000	0.000	1.800
input 1	8.600	0.000	0.000	8.600
input 2	1.800	0.000	0.000	1.800

LISTING OF PEERS:

peer	*lambda weight*
1	1.000

Results for firm: 2
Technical efficiency = 0.434

PROJECTION SUMMARY:

variable	*original value*	*radial movement*	*slack movement*	*projected value*
output 1	0.200	0.000	0.000	0.200
input 1	2.200	–1.244	0.000	0.956
input 2	1.700	–0.962	–0.538	0.200

LISTING OF PEERS:

peer	*lambda weight*
1	0.111

Results for firm: 3
Technical efficiency = 1.000

PROJECTION SUMMARY:

variable	*original value*	*radial movement*	*slack movement*	*projected value*
output 1	2.800	0.000	0.000	2.800
input 1	15.600	0.000	0.000	15.600
input 2	2.600	0.000	0.000	2.600

LISTING OF PEERS:

peer	*lambda weight*
3	1.000

Results for firm: 4
Technical efficiency = 0.620

PROJECTION SUMMARY:

variable	*original value*	*radial movement*	*slack movement*	*projected value*
output 1	4.100	0.000	0.000	4.100
input 1	31.600	–12.011	0.000	19.589
input 2	12.300	–4.675	–3.525	4.100

LISTING OF PEERS:

peer	*lambda weight*
1	2.278

5.4.3 Using GAMS for DEA Computations

General Algebraic Modelling System (GAMS) is a high-level modelling system for mathematical programming problems. GAMS modelling language has been used in a variety of linear, non-linear, and mixed-integer programming models, general equilibrium models, and network models. It has a syntax similar to that of a high-level computer programming language. The GAMS system is especially useful for large, complex problems. GAMS is available for use on systems ranging from personal computers to supercomputers. GAMS allows the user to concentrate on modelling the problem by making the setup simple. It is used in conjunction with a solver. Further information on GAMS can be obtained at *www.gams.com*.

As the DEA technique is based on linear programming, it can be implemented on GAMS. But, GAMS, in its original form, has not been designed for the repeated LP computations needed for DEA. However, latest versions of GAMS provide the flexibility to include DEA-type computations. For example, DEA computations can be performed by invoking a new interface, GAMS/DEA interface, in a GAMS program. Please see the GAMS Web site for further details.

Some of the DEA studies reported in the literature have been carried out using GAMS. Olesen and Petersen (1996) have provided GAMS programming code for DEA. Based on this, Walden and Kirkley (2000) have developed several GAMS programs for modelling production efficiency and fishing capacity in an application of DEA in marine fisheries. The study by Productivity Commission (1999) has used DEA for assessing the performance of Australian Railways. A copy of the GAMS programming code employed for the study is available at *http://www.pc.gov.au/inquiry/rail/finalreport/supplement/gamscode.doc*.

5.4.4 Using Spreadsheets (e.g., MS Excel) for DEA Computations

Over the last several years, the computational capabilities of spreadsheet packages have been growing rapidly. Many commercial versions of spreadsheet software provide special tools for performing many statistical analysis, optimization and simulation tasks. For example, Microsoft's spreadsheet package Microsoft Excel can perform many complex computations using special add-ins such as Solver and Data Analysis. More complex computations can be performed using the macro features of spreadsheet packages. For Excel, the macro programming language, Visual Basic for Applications (VBA) can be used.

The flexibility and computational superiority offered by spreadsheet packages can be exploited advantageously to perform DEA computations. It is necessary to enter all the input and output data required for a DEA study. We have seen that the DEA score of a particular DMU can be obtained using linear programming. This can be easily solved in Excel by calling the Solver add-in. Textbooks on operations research (e.g., Anderson et al. 2000) provide further help in using Microsoft Excel for solving linear programming problems. In this section, the use of Excel for solving an output maximizing multiplier DEA program for computing efficiency of Firm A (see Chapter 2) will be illustrated. Please note that the description is limited to illustrating the use of Solver for solving a DEA program. For a complete description of the options available in Excel Solver, the reader should consult its user manual.

In order to enable Excel or any other spreadsheet program to solve a DEA problem, it is important to provide all the data—the decision variables, the objective function, the left-hand sides of the constraints, and the right-hand sides of the constraints.

5.4.4.1 Spreadsheet formulation: Follow these steps to solve any problem using spreadsheets.

Step 1: Enter the basic data as a separate part, usually at the top of the spreadsheet or as a separate worksheet in a workbook. In the example given below, the basic input–output data for the DEA program is entered at the top of the spreadsheet, from the rows 4 to 8 (from Cell A4 to Cell D8). Captions are provided in

Row 4, and each of the other rows corresponds to the data of Firms A, B, C and D, respectively. For example, Cell B5 represents the capital employed for Firm A, Cell C5 represents the number of employees for Firm A and Cell D5 represents the value added for Firm A. The spreadsheet formulation for the output maximizing multiplier DEA program for computing the efficiency of Firm A is shown in Figure 5.2.

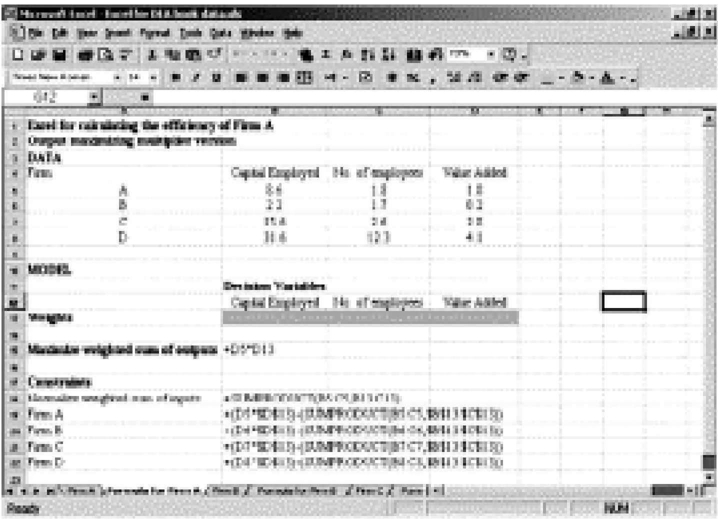


Figure 5.2 Spreadsheet (with formulae) for solving the output maximizing multiplier DEA program for Firm A

Step 2: Specify cell locations for all the decision variables.

The decision variables used in the example are the weights of inputs (capital employed, CAP and number of employees, EMP) and output (value added, VA). Cell B13 is reserved for the weight of CAP, Cell C13 for the weight of EMP, and Cell D13 for the weight of VA. These cells are highlighted in Figure 5.2. Note that no values are selected initially for these cells. Excel's Solver, when invoked, computes the optimal values of these weights that optimize the objective function.

Step 3: Select a cell and enter the formula for computing the value of the objective function.

The objective function of this example is the product of the weight of VA and the value of VA for Firm A. This formula is entered in Cell B15. Note that, for the sake of explanation, the spreadsheet shows the formula descriptively. Actually, in a spreadsheet, when a formula is entered, only the result of the computation is shown in the cell; the formula cannot be seen. In Excel, formulae should start either with a + sign or = sign; otherwise, Excel does not recognize the formula and considers whatever is typed to be a set of text information.

Step 4: Select a cell and enter the formula for computing the left-hand side of each constraint.

The first constraint is the normalizing constraint that equates the weighted sum of inputs to unity. This formula is entered using a special function of Excel called SUMPRODUCT. The formula,

$$+\,\text{SUMPRODUCT (B5:C5, B13:C13)}$$

is equivalent to

$$+\,(\text{B5} * \text{B13}) + (\text{C5} * \text{C13}).$$

Step 5: Select a cell and enter the formula for computing the right-hand side of each constraint.

This step is necessary if, in a linear program, the right-hand side of constraints have to be computed using some formulae. However, for the DEA programs, this step is normally not necessary as the right-hand sides of the constraints are either 1 (for normalizing constraint) or 0 (for other constraints). They can be directly entered in Excel's **Solver Parameters** box, and will be described shortly.

5.4.4.2 Spreadsheet solution: Once the spreadsheet formulation is complete, the optimal solution can be obtained by accessing Excel's Solver add-in. Excel Solver is a general optimization tool, and is not specifically designed for DEA computations. The following steps are necessary to solve the DEA problem presented in Figure 5.2.

Step 1: Select the **Tools** pull-down menu.

Step 2: Select the **Solver** option. (If the **Solver** option is not available, the Solver Add-in has to be loaded by invoking the **Tools** menu and choosing the **Add-Ins...** option. This is a one-time procedure and need not be carried out every time Solver is invoked.) The **Solver Parameters** dialogue box appears as shown in Figure 5.3.

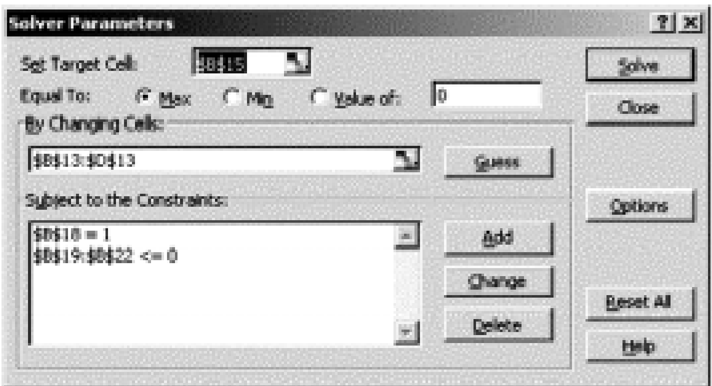


Figure 5.3 Solver Parameters dialogue box

Step 3: Complete the entries as shown in Figure 5.3 in the **Solver Parameters** dialogue box. The target cell is the cell that has the formula of the objective function (Cell B15, in this example). As the problem requires the maximization of this objective function, the radio button **max** has to be chosen. The decision variable cells (Cells B13 to D13, specified in Excel as B13:C13) have to entered in the **By Changing Cells** box.

Constraints should be specified by clicking the **Add** button. This will bring the **Add Constraint** dialogue box, shown in Figure 5.4.

The first constraint can be specified as shown in Figure 5.4. The formula for the left-hand side of this constraint has already been entered in Cell B18; hence this cell reference has to be specified below the **Cell Reference** box. As this is an equality constraint, the equality symbol has to be chosen in the next box. The right-hand side of this constraint is 1; hence this number is entered in

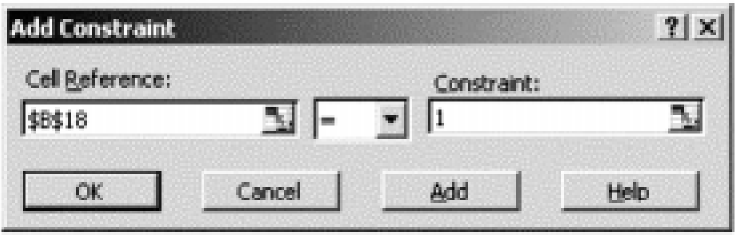


Figure 5.4 Add Constraints dialogue box

the **Constraint** box. Clicking the **OK** button will enter this constraint in the **Solver Parameters** dialogue box. The **Solver Parameters** dialogue box reappears.

The remaining constraints have a similar structure: their right-hand sides have similar formulae and have been entered in Cells B19 to B22; all of them are less-than-or-equal-to constraints; and their right-hand sides are zero. All of them can be entered in the **Add Constraint** dialogue box. The **Cell Reference** should be B19:B22, the symbol **£** should be chosen in the next box, while the value 0 should be entered in **Constraint** box. Note that the constraints can also be entered one at a time.

Step 4: When the **Solver Parameters** dialogue box reappears, click the **Options** button. This will bring up the **Solver Options** dialogue box shown in Figure 5.5. Select the radio buttons corresponding to **Assume Linear Model** and **Assume Non-Negative** as shown in the figure. This action specifies non-negativity constraints. Normally, these are the only two data to be provided in this dialogue box, but, if necessary, other data (say, precision) may also be changed.

Step 5: When the **Solver Parameters** dialogue box reappears, click the **Solve** button. This will bring up the **Solver Results** dialogue box, shown in Figure 5.6. If all the previous data has been entered correctly, the Solver program would have found a solution. Click the radio button for **Keep Solver Solution**, and click **OK** to produce the optimal solution output. This output is shown in Figure 5.7.

According to the optimal solution, the optimal objective function value, available in Cell B15, is equal to 1 signifying that Firm A is an efficient one. For inefficient firms, this optimal objective

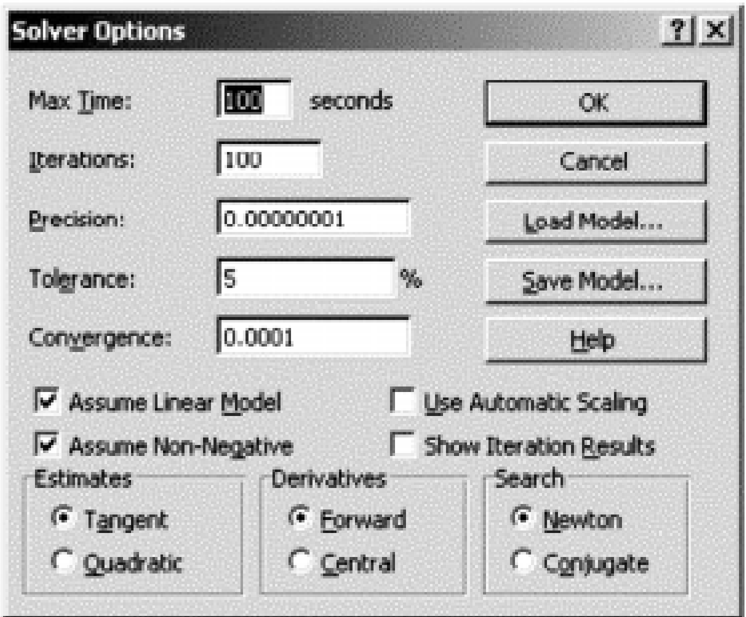


Figure 5.5 Solver Options dialogue box

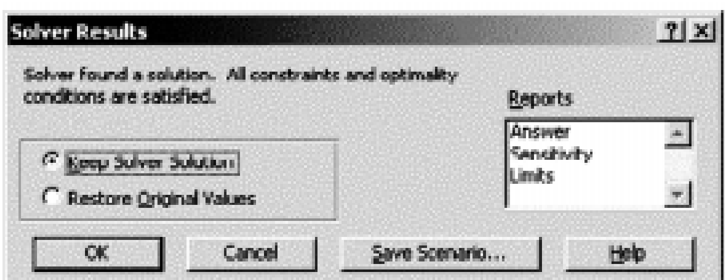


Figure 5.6 Solver Results dialogue box

function value will be less than 1. The optimal weights of the inputs and outputs can be obtained from the Decision Variables Cells (B13 to D13).

Note that the above procedure solves the DEA program for Firm A and hence calculates DEA efficiency for Firm A only. To

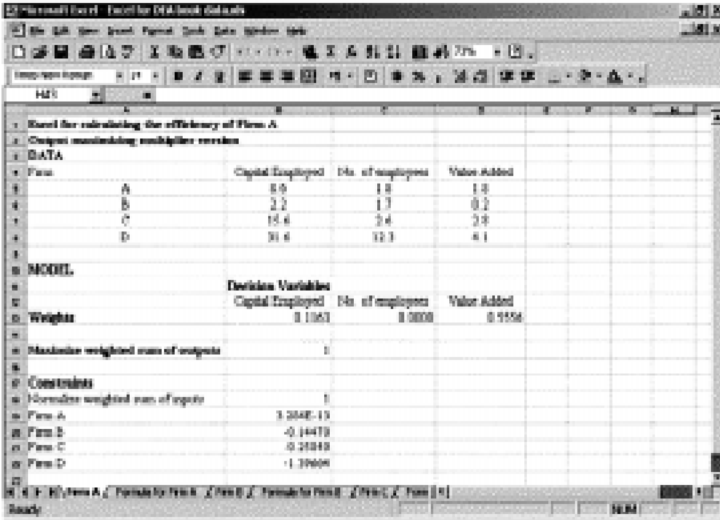


Figure 5.7 Solver solution for the output maximizing multiplier DEA program for Firm A

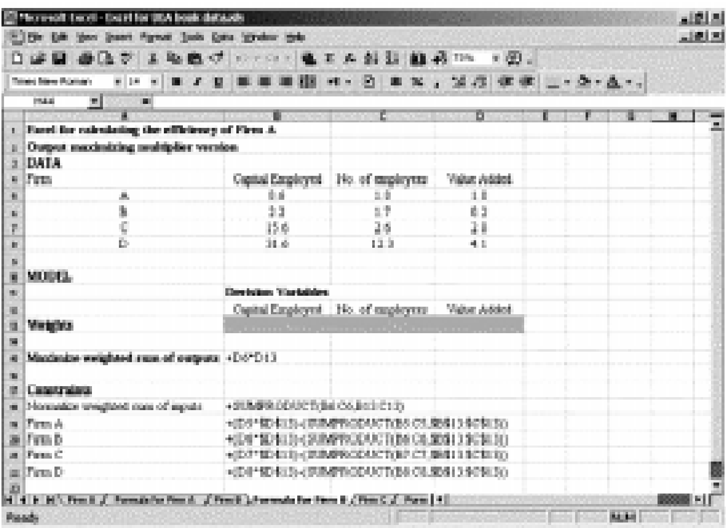


Figure 5.8 Spreadsheet (with formulae) for solving the output maximizing multiplier DEA program for Firm B

get the efficiency score for any other firm, say Firm B, this procedure has to be repeated using a new objective function and normalization constraint. The spreadsheet (with formulae) for computing the efficiency of Firm B is shown in Figure 5.8.

In order to get the DEA efficiency score for all DMUs, Excel Solver should be used to solve the DEA linear programming problem repeatedly for each DMU. This means that the objective function and the normalization constraint have to be modified for each DEA computation. This can be tedious, especially when the number of DMUs are of the order of a few hundreds. This process can be automated using a macro. Since VBA can be used to control Solver, macros can be written using VBA. Macro programming is beyond the scope of this book. Proudlove (2000) has provided one such VBA macro for repeatedly solving LP programs for DEA computations. Others who have used spreadsheets for performing DEA computations include Jablonský (1999) and Premachandra et al. (1998). The DEAFrontier software bundled with Zhu (2002), is based on an Excel Add-in for DEA computations.

6

DEA Bibliography and Applications

Data Envelopment Analysis (DEA) is an actively growing field of operations research and performance measurement. Since 1978, when the first paper on DEA was published, over 1,000 articles, books and dissertations have appeared. The original CCR and BCC formulations of DEA have been extended to include more complications involving categorical variables, discretionary and non-discretionary variables, incorporating value judgements, weight restrictions, Malmquist productivity indices, technical change in DEA, and many other topics. Some of extension have been discussed in the earlier chapters of this book. The application fields of DEA are also diverse: from education (schools and universities) to banks, health care (hospitals, clinics), prisons, agricultural production, banking, transportation, courts, and to many others. Hence, the bibliography of DEA is a constantly growing one. Seiford (1994, 1996) presented one of the recent bibliographies on DEA. The bibliography includes more than 472 published articles and dissertations related to DEA, covering the period 1978–92. Professor Seiford maintains a Web site on DEA at the Internet site, *http://www.ecs.umass.edu/mie/dea/* as on May 2001, where a comprehensive DEA bibliography is also available. Professor Tim Anderson, Portland State University, USA, is maintaining a DEA bibliography at the Internet site, *http://www.emp.pdx.edu/dea/deabib.html*. A very comprehensive DEA bibliography is maintained at the University of Warwick by Professor Emrouznejad (1995–2001). This bibliography is available on the Internet at the site, *http://www.deazone.com/*. Abstracts of many of the articles mentioned in the bibliography are also available in the Web site. The bibliography is available in printed form on request.

Many of the Web sites mentioned in the previous chapters also provided some bibliographic information on DEA. There are several other bibliographies on the Internet, which can be accessed via the links available from the sites mentioned above.

6.1 Brief Literature Survey

In this section, a brief review of the applications of DEA is provided. This review is by no means comprehensive, but is intended to provide references to selected publications in the application areas mentioned in the following paragraphs.

Seiford and Thrall (1990), Charnes et al. (1994) and Seiford (1996) have provided detailed accounts of the important developments in the history of DEA. Golany and Roll (1989) have given broad guidelines for describing the applications of DEA.

The application areas of DEA include assessing social and economic performances of countries and cities of the world, productivity assessment in the public sector, and evaluation of school performance. Golany and Thore (1997) have compared different nations in terms of their overall socio-economic performance using DEA. They used the ratio of real domestic investment to real GDP, ratio of real government consumption expenditure to real GDP, and the ratio of government expenditure on education to nominal GDP as inputs. The growth rate of per capita GDP, one minus infant mortality rate, enrolment ratio for secondary education, and ratio of nominal social insurance and welfare payments to nominal GDP were considered as outputs.

DEA has found a number of applications for productivity assessment, especially in the public sector. Ganley and Cubbin (1992) have described the applications of DEA to schools and prisons. Yeh (1996) has reported an application of DEA to banks. An application to the health sector is available in Bates et al. (1996). Fare et al. (1996) have described an application of DEA in the environmental performance evaluation for the case of fossil-fuel-fired electric utilities. DEA has also been applied to the transport sector, for example, to measure transportation productivity by Good and Rhodes (1990) for airlines, and by Oum and Yu (1991) for railways. Adolphson et al. (1989) have used DEA for railroad property valuation. Hjalmarsson and Odeck (1996) have evaluated the efficiency of trucks in road construction and road maintenance. Nozick et al. (1998) have used DEA to evaluate travel demand measures and programs.

There is a vast volume of literature on DEA applications to school performance evaluation. Perhaps the oldest application was made in USA (Bessent and Bessent 1980) for evaluating the performance of 55 schools in an urban school district with 60,000 pupils. Subsequently, the same group (Bessent et al. 1982; 1984) conducted more studies in the American context. Ray (1991) has applied DEA to estimate relative efficiencies in the public school district of Connecticut. He has further analysed the effect of several socio-economic factors using regression. DEA has been applied to schools in UK (Smith and Mayston 1987) and Denmark (Olesen and Petersen 1995).

A complete list of all articles referred to in this book is available at the end of this book.

6.2 Selected DEA Applications

From the time the DEA technique was proposed more than two decades ago, the methodology has been used for numerous traditional as well as novel applications. Several reviews of DEA have been published regularly in journals and books. It is not the intention here to present an exhaustive overview of all the applications of DEA. In the sub-sections below, the use of DEA for some traditional and novel applications, mainly from the articles published by the author, are described in detail. For most of the studies presented below, the software package from the University of Warwick (Windows version 1.03) has been used for carrying out the DEA computations.

6.2.1 Productivity Assessment of State Transport Undertakings in India

A straightforward application of DEA to assess the productivity of 29 State Transport Undertakings (STUs) during the year

1993–94 is presented by Ramanathan (1999). The data used for productivity assessment has been obtained from the information published regularly by the Association of State Road Transport Undertakings (e.g., CIRT 1995). The public bus operators considered in this study are listed in Table 6.1, along with geographical information.

Three inputs have been considered as most important in producing the output. They include fleet size, representative of the capital input; total number of staff, representative of the labour input; and diesel consumption, a dominant material input. Table 6.2 lists the inputs.

Two demand-side measures (passengers carried and passenger kilometres effected) and two supply-side measures (seat kilometres and vehicle seat hours) of output, at the least, can be identified from a study of the literature. The supply side output measures may be considered alternatively as intermediate outputs because not all of them may be used for service. Hence, demand-oriented output measures are considered for the analysis here. More specifically, being a composite measure of the numbers of passengers carried and the average lead travelled by passengers, passenger kilometres has been chosen as a single measure of output for the bus operators. Details are listed in Table 6.2.

The DEA analysis has been carried out using both CRS and VRS assumptions. The efficiency scores of the STUs are shown in Table 6.3 along with their ranks.

Table 6.3 shows that there is a high degree of correlation between the efficiencies under CRS and VRS assumptions. The correlation coefficient is 0.83. As expected, the VRS scores are higher than the CRS scores. The average efficiency score under CRS assumption is about 75 per cent. Fourteen STUs have a score lower than this average. The average efficiency score under VRS assumption is about 84 per cent, with 10 STUs placed below this average figure.

A striking observation in the efficiency ratings based on CRS assumption is that the STUs operated as companies have received the highest ratings. However, though companies topped the list in terms of the VRS efficiency score, their domination is not so perfect. This might mean that STUs that are not operated as companies did not receive higher CRS efficiency scores because they did not operate at their most productive scale size.

Table 6.1 State Transport Undertakings in India

Acronym	*Name*	*Nature of Service*	*State of Operation*	*Nature of the Organization*
AMTS	Ahmedabad Municipal Transport Service	Urban	Ahmedabad City	Municipal Undertaking
APSRTC	Andhra Pradesh State Road Transport Corporation	Mofussil	Andhra Pradesh	Corporation
ASTC	Annai Satya Transport Corporation Limited	Mofussil	Tamil Nadu	Company
BEST	Bombay Electric Supply & Transport Undertaking	Urban	Mumbai City	Municipal Undertaking
CRC	Cholan Roadways Corporation Limited	Mofussil	Tamil Nadu	Company
CTC	Cheran Transport Corporation Limited	Mofussil	Tamil Nadu	Company
DTC	Delhi Transport Corporation	Urban	Delhi	Corporation
GSRTC	Gujarat State Road Transport Corporation	Mofussil	Gujarat	Corporation
HRTC	Himachal Road Transport Corporation	Hilly	Himachal Pradesh	Corporation
KMTU	Kohlapur Municipal Transport Undertaking	Urban	Kohlapur City	Municipal Undertaking
KnSRTC	Karnataka State Road Transport Corporation	Mofussil	Karnataka	Corporation
MGRTC	Puratchi Thalaivar MGR Transport Corporation Limited	Mofussil	Tamil Nadu	Company
MPTC	Marudhu Pandiyar Transport Corporation Limited	Mofussil	Tamil Nadu	Company

Acronym	*Name*	*Nature of Service*	*State of Operation*	*Nature of the Organization*
MSRTC	Maharashtra State Road Transport Corporation	Mofussil	Maharashtra	Corporation
NBSTC	North Bengal State Transport Corporation	Mofussil	West Bengal	Corporation
NGST	Nagaland State Transport	Hilly	Nagaland	Government Department
NTC	Nesamony Transport Corporation Limited	Mofussil	Tamil Nadu	Company
OSRTC	Orissa State Road Transport Corporation	Mofussil	Orissa	Corporation
PATC	Pattukkottai Azhagiri Transport Corporation Limited	Mofussil	Tamil Nadu	Company
PRC	Pandiyan Roadways Corporation Limited	Mofussil	Tamil Nadu	Company
PTC	Pallavan Transport Corporation Limited	Urban	Chennai City	Company
RMTC	Rani Mangammal Transport Corporation Limited	Mofussil	Tamil Nadu	Company
RSRTC	Rajasthan State Road Transport Corporation	Mofussil	Rajasthan	Corporation
SKNT	Sikkim Nationalized Transport	Hilly	Sikkim	Government Department
STPJB	State Transport Punjab	Mofussil	Punjab	Government Department
TPTC	Thanthai Periyar Transport Corporation Limited	Mofussil	Tamil Nadu	Company
TRPTC	Tripura Road Transport Corporation	Hilly	Tripura	Corporation
TTC	Thiruvalluvar Transport Corporation Limited	Mofussil	Tamil Nadu	Company
UPSRTC	Uttar Pradesh State Road Transport Corporation	Mofussil	Uttar Pradesh	Corporation

Table 6.2 Inputs and outputs of state transport undertakings

Unit	*Fleet Size (no.)*	*Total Staff (no.)*	*Diesel Consumption (kilolitre)*	*Passenger Kilometres (million)*
MSRTC	15,483	1,11,979	3,17,679	53,409
APSRTC	15,235	1,19,630	3,28,010	64,486
KnSRTC	9,899	63,712	2,16,159	37,460
GSRTC	8,945	59,706	1,86,557	29,485
UPSRTC	8,023	56,864	1,44,478	24,053
RSRTC	4,115	25,892	90,268	15,903
STPJB	2,368	12,551	54,411	7,633
CTC	1,290	8,338	42,117	8,147
PRC	1,150	8,992	34,971	6,939
TTC	940	8,733	49,813	7,559
NBSTC	947	6,707	19,637	2,500
OSRTC	954	7,223	18,244	2,435
TPTC	809	6,078	26,776	5,596
CRC	766	6,069	25,134	4,804
PATC	700	5,092	23,372	5,062
MPTC	700	5,268	23,394	4,819
MGRTC	661	4,902	20,220	4,411
RMTC	644	4,668	21,545	4,426
NTC	593	4,470	18,326	3,603
ASTC	584	4,116	19,163	4,394
HRTC	1,598	9,179	35,830	3,775
NGST	198	1,045	1,432	143
SKNT	198	1,114	1,030	57
TRPTC	130	802	647	73
DTC	3,706	34,220	69,710	11,722
BEST	3,067	38,965	81,009	11,060
PTC	2,094	19,244	45,225	10,131
AMTS	753	6,105	11,815	1,795
KMTU	117	1,291	2,652	417

It is necessary to test the validity of the CRS or VRS assumption before we analyse further the performance of STUs using DEA. The regression exercise to be described later has shown that the efficiency scores (CRS or VRS) are not influenced significantly by the number of routes operated by a STU (considered as a proxy for the volume of operation). Hence, only the efficiency scores

Table 6.3 DEA efficiency scores and ranks of STUs under constant returns to scale (CRS) assumption and under variable returns to scale (VRS) assumption

Unit	*CRS*	*Rank*	*VRS*	*Rank*
AMTS	66.26	21	67.00	24
APSRTC	85.73	11	100.00	1
ASTC	100.00	1	100.00	1
BEST	59.54	23	71.52	23
CRC	83.35	13	89.38	16
CTC	91.52	7	100.00	1
DTC	73.33	16	79.01	21
GSRTC	68.92	19	85.08	19
HRTC	45.94	27	50.69	28
KMTU	68.54	20	100.00	1
KnSRTC	75.57	15	100.00	1
MGRTC	95.12	5	95.32	12
MPTC	91.31	8	95.17	13
MSRTC	73.32	17	87.86	17
NBSTC	55.53	25	55.58	27
NGST	43.45	28	55.66	26
NTC	85.73	11	85.80	18
OSRTC	58.20	24	58.25	25
PATC	95.93	4	100.00	1
PRC	86.53	10	97.08	11
PTC	97.69	3	100.00	1
RMTC	91.14	9	93.13	14
RSRTC	76.83	14	92.69	15
SKNT	24.13	29	35.06	29
STPJB	61.18	22	73.83	22
TPTC	91.84	6	99.46	10
TRPTC	49.30	26	100.00	1
TTC	100.00	1	100.00	1
UPSRTC	72.60	18	82.35	20

under the CRS assumption have been considered in the DEA analysis hereafter.

This DEA analysis also provides important information about the sources of inefficiencies of those STUs that do not have unit efficiencies. Obviously, the sources are identified by comparing the inefficient STU with the efficient ones (performance targets). The targets as well as the reduction or improvement required in the performance of STUs to achieve the targets are listed Table 6.4. In the case of inputs, the target specifies the amount the STU in question ought to have consumed if it were to be rated efficient. Obviously, an inefficient STU would have consumed more inputs than the target. The extent of reduction needed is also listed in the table. Similarly, in the case of outputs, the target specifies the amount required to be achieved if the STU is to be rated efficient; the extent of improvement needed is also listed for each STU.

Both ASTC and TTC have been rated as efficient under the CRS assumption. Hence, their targets coincide with the actual achievements, and the extent of reduction of inputs or improvement in their output is 0. In the case of other STUs, the targets are different from their actual performances. For example, the fleet size of AMTS has to be reduced from its present size of 743 to about 360 (that is by about 52 per cent) in order for it to qualify as an efficient STU. The STU with the lowest ranking, namely SKNT, requires significant reductions (more than 80 per cent) in fleet size or staff, or a very significant (more than 300 per cent) improvement in output to be rated efficient. Note that most STUs are using their fuel efficiently. But, Table 6.4 shows that many of the inefficient STUs have excess capacities in terms of fleet size and staff. Of the two factors, overstaffing is more pronounced, which is a general characteristic of public sector organizations.

An improved level of fleet utilization is necessary for the STUs to be more efficient. This points to the need for efficient management of available non-human resources. If the STUs achieve higher output (passenger kilometres) using the same amount of input, their efficiency score would improve. The extent of improvement needed in the output of each STU is available in Table 6.4.

The inefficient STUs would do well if they utilize their human capital properly through better human resource management. This can be achieved through proper co-ordination, and motivation of the employees for better performance.

6.2.1.1 Identification of the effect of uncontrollable factors on the efficiency scores of STUs: The efficiency scores obtained using the DEA model described so far should be more appropriately termed *gross* efficiency scores because they include the effects of certain variations in the inputs and outputs which are

Table 6.4 Targets (in original units mentioned in Table 6.2) and reduction/improvement needed (per cent)

Unit	*Fleet Size*		*No. of Staff*		*Fuel Consumption*		*Passenger Kilometres*	
	Target	*Reduction*	*Target*	*Reduction*	*Target*	*Reduction*	*Target*	*Improvement*
AMTS	360	52.18	2,538	58.43	11,815	0.00	27,094	50.91
APSRTC	9,996	34.39	70,454	41.11	3,28,010	0.00	7,52,162	16.64
ASTC	584	0.00	4,116	0.00	19,163	0.00	43,942	0.00
BEST	2,469	19.50	17,400	55.34	81,009	0.00	1,85,761	67.96
CRC	766	0.00	5,399	11.05	25,134	0.00	57,635	19.98
CTC	1,183	8.29	8,338	0.00	38,819	7.83	89,016	9.26
DTC	2,125	42.67	14,973	56.24	69,710	0.00	1,59,852	36.37
GSRTC	5,686	36.44	40,071	32.89	1,86,557	0.00	4,27,796	45.09
HRTC	1,092	31.67	7,696	16.16	35,830	0.00	82,162	117.68
KMTU	81	30.94	570	55.88	2,652	0.00	6,081	45.89
KNSRTC	6,588	33.45	46,429	27.13	2,16,159	0.00	4,95,676	32.32
MGRTC	616	6.78	4,343	11.40	20,220	0.00	46,367	5.13
MPTC	700	0.00	4,981	5.45	23,394	0.00	52,779	9.52
MSRTC	9,682	37.47	68,235	39.06	3,17,679	0.00	7,28,471	36.40
NBSTC	599	36.80	4,218	37.11	19,637	0.00	45,031	80.09
NGST	44	77.98	308	70.56	1,432	0.00	3,284	130.13
NTC	559	5.82	3,936	11.94	18,326	0.00	42,024	16.64
OSRTC	556	41.72	3,919	45.75	18,244	0.00	41,835	71.82
PATC	700	0.00	4,978	2.23	23,372	0.00	52,774	4.25
PRC	1,066	7.32	7,512	16.46	34,971	0.00	80,192	15.56
PTC	1,378	34.18	9,714	49.52	45,225	0.00	1,03,706	2.37

(*Table 6.4 contd.*)

(Table 6.4 contd.)

Unit	*Fleet Size*		*No. of Staff*		*Fuel Consumption*		*Passenger Kilometres*	
	Target	*Reduction*	*Target*	*Reduction*	*Target*	*Reduction*	*Target*	*Improvement*
RMTC	644	0.00	4,585	1.78	21,545	0.00	48,563	9.72
RSRTC	2,751	33.15	19,389	25.12	90,268	0.00	2,06,995	30.16
SKNT	31	84.14	221	80.13	1,030	0.00	2,363	314.49
STPJB	1,658	29.97	11,687	6.88	54,411	0.00	1,24,770	63.45
TPTC	809	0.00	5,727	5.77	26,776	0.00	60,931	8.88
TRPTC	20	84.85	139	82.67	647	0.00	1,484	102.82
TTC	940	0.00	8,733	0.00	49,813	0.00	75,589	0.00
UPSRTC	4,403	45.12	31,033	45.43	1,44,478	0.00	3,31,303	37.74

beyond the control of individual STUs. For example, an STU with an older fleet will not be able to perform as well as the firms with younger fleets. Similarly, if an STU operates its bus services in hilly terrain, it is likely to consume more input for a given output. If the factors beyond managerial control are not considered, any inference regarding the efficiency of STUs is likely to be misleading.

As discussed in Chapter 4, DEA studies in the literature have recognized this problem, and have used different approaches to overcome it. For example, Majumdar (1997) has employed regression analysis to study the effect of several factors on the efficiency scores obtained for the telecommunication industry in USA. Ray (1991) has used a similar approach while comparing the efficiency of some schools in USA. Yeh (1996) has employed the Principal Factor Analysis approach in his application of DEA for bank performance evaluation. Hence, a regression-based approach is employed here to study the effect on the DEA scores of those factors which are beyond managerial control but are likely to affect the performance of STUs.

Several factors, including as the following, have been considered for the regression analysis.

(*a*) Number of routes operated by an STU. This variable is a proxy for the volume of operation of an STU.
(*b*) Age of fleet (denoted as AGE). STUs with older fleets are likely to be less efficient compared to those with relatively younger fleets.
(*c*) Share of urban operations expressed as a percentage (CITY). Some STUs operate exclusively in urban areas, some operate exclusively in rural areas, while others operate in both the areas. Urban services are likely to be more efficient because of higher capacity utilization.
(*d*) The nature of the terrain. The STUs operating on a hilly terrain are likely to incur more expenditure, especially for fuel, than those operating on a plain terrain. A dummy variable, denoted as HILL, is used to represent hill services.
(*e*) The average number of passenger per bus per day (PASS-DENSITY). This is likely to indicate the capacity utilization of a bus.

In addition to the above factors, several others, including deficit per passenger, ratio of labour to other inputs, average lead, etc., have also been used in the regression exercise. However, their inclusion did not yield satisfactory results.

The first striking result emerging from the regression exercise is the low statistical significance associated with the variable representing the number of routes. The same result has also been obtained with regard to the efficiency scores under the VRS assumption. This supports the view that the efficiency scores are not affected by the volume of operation. All the other variables listed above yielded good results. Different combinations of these variables along with linear and logarithmic functional forms have been employed to generate four regression models. The results are given in Table 6.5.

Table 6.5 Results of the regression exercise with the efficiency score (under CRS assumption) as the dependent variable

Regressor	*Linear*		*Logarithmic*	
	Model 1	*Model 2*	*Model 3*	*Model 4*
CONSTANT	97.221*	88.677*	5.002*	4.216*
	(17.886)	(14.851)	(42.620)	(15.387)
AGE	–4.538*	–3.831*	–0.422*	–0.383*
	(–4.152)	(–4.064)	(–5.752)	(–4.419)
HILL	–26.724*	–23.914*	–0.415*	–0.349**
	(–3.860)	(–3.449)	(–5.036)	(–2.763)
CITY	0.175**	–	0.028*	–
	(2.443)	–	(4.747)	–
PASSDENSITY	–	0.014*	–	0.117*
	–	(2.921)	–	(2.865)
R^2 adjusted	0.680	0.705	0.839	0.770

Notes: *t*-values are shown in parentheses. Symbols *, ** indicate that the corresponding coefficients are significant at 1 per cent and 5 per cent levels.

Note that AGE and HILL have been found to be highly significant in all the models consistently. They have the expected negative values, which signifies that the efficiency score would decrease if the average age of fleet of a STU increases or if the STU operates on a hilly terrain. The vairable CITY has positive values, indicating an increase in efficiency for urban operations. Similarly, the variable PASSDENSITY also has the expected positive values.

6.2.2 Comparative Performance of Schools

Another simple application of DEA that evaluates the performance of some schools in the Netherlands has been described by Lootsma and Ramanathan (1999) and Ramanathan (2001a).

The Ministry of Education, the Netherlands, has recently introduced an assessment of secondary schools in the country on the basis of the performance of the pupils (Regional Guidebook 1998). The leading criteria for the assessment of a school in the current year are the following.

- The percentage of pupils who successfully proceeded without delay, from the start of the third schoolyear until the final diploma at the end of the sixth year, NO-DELAY;
- the percentage of pupils who were successfully examined in more than the prescribed minimum of seven subjects, EXTRA-SUBJ; and
- the average grade obtained by the pupils in the subjects which were centrally examined, i.e, nationwide, per subject on the same day, and with identical exercises, AVG-GRADE.

The results of the assessment, summarized on so-called quality cards, are available on request. The quality cards, one per school, are bundled into regional guidebooks so that each school is informed about the performance of the neighbouring schools.

The basic data compiled from the Regional Guidebook (1998) on grammar schools and the grammar streams of comprehensive schools, totalling 46, in the region containing The Hague, Leiden, Delft and Zoetermeer is given in Table 6.6. Data regarding the performance of these schools based on the three outputs (criteria) is presented in the table. One can differentiate two kinds of inputs for these schools—discretionary and non-discretionary. Discretionary inputs are normally under the control of the school management; they include the number of teachers, pupil–teacher ratio, total cost, and cost per pupil. It is believed that these inputs are approximately the same in the Netherlands. However, the non-discretionary inputs (i.e., inputs which are beyond the control of

Table 6.6 Basic data for the quality assessment of some secondary schools in the Netherlands

Location and Card No.	*Location*	*Category*[a]	*No. of Pupils*[b]	*Grammar School*[c]	*NO-DELAY (%)*	*EXTRA-SUBJ (%)*	*AVG-GRADE*
A1	Alphen a/d Rijn	1	220	0	48	14	6.0
A3	Alphen a/d Rijn	0	193	0	53	10	6.1
A5	Alphen a/d Rijn	1	592	0	63	16	6.6
D6	Delft	1	619	0	66	23	6.8
D10	Delft	1	476	0	56	25	6.5
D11	Delft	0	261	0	47	32	6.3
K14	Katwijk	1	258	0	34	11	6.6
L16	Leiden	0	1,035	1	76	65	6.9
L17	Leiden	1	208	0	45	48	6.8
L18	Leiden	0	136	0	46	52	5.8
L19	Leiden	1	533	0	54	9	6.7
L23	Leiden	0	254	0	53	0	6.4
L27	Leiden	1	726	0	50	13	6.7
LD28	Leidschendam	1	158	0	59	43	6.4
N31	Naaldwijk	1	353	0	48	10	6.6
N34	Naaldwijk	1	315	0	55	0	6.3
NR36	Noordwijk	0	193	0	47	9	6.6
O37	Oegstgeest	2	630	0	64	16	6.5
R38	Rijswijk	1	232	0	52	25	6.3
R40	Rijswijk	0	239	0	57	30	6.2
H48	The Hague	1	246	0	62	28	6.5

Location and Card No.	Location	Category[a]	No. of Pupils[b]	Grammar School[c]	NO-DELAY (%)	EXTRA-SUBJ (%)	AVG-GRADE
H52	The Hague	0	617	1	77	57	7.3
H53	The Hague	1	509	1	79	63	7.0
H54	The Hague	1	121	0	35	7	6.7
H55	The Hague	1	84	0	30	25	6.1
H57	The Hague	0	93	0	34	0	5.6
H60	The Hague	0	573	0	33	19	6.3
H62	The Hague	1	236	0	41	0	6.4
H66	The Hague	0	278	0	53	0	6.2
H67	The Hague	1	63	0	46	20	6.5
H68	The Hague	2	495	0	49	26	6.2
H69	The Hague	1	261	0	39	15	6.6
H70	The Hague	2	190	0	54	21	6.6
H74	The Hague	0	467	0	38	8	6.4
H75	The Hague	1	313	0	46	30	6.6
S76	Sassenheim	2	187	0	69	6	6.3
V77	Voorburg	2	264	0	41	7	6.0
V79	Voorburg	1	421	0	43	5	6.2
V80	Voorburg	2	173	0	54	29	6.0
VH82	Voorhout	1	118	0	50	17	6.8

(Table 6.6 contd.)

(*Table 6.6 contd.*)

Location and Card No.	*Location*	*Category*[a]	*No. of Pupils*[b]	*Grammar School*[c]	*NO-DELAY (%)*	*EXTRA-SUBJ (%)*	*AVG-GRADE*
W85	Wassenaar	1	406	0	54	33	6.6
W86	Wassenaar	2	437	0	62	42	6.3
Z87	Zoetermeer	0	130	0	20	29	5.8
Z91	Zoetermeer	1	327	0	61	5	6.3
Z92	Zoetermeer	1	245	0	56	52	6.5
Z94	Zoetermeer	1	465	0	70	18	6.6

Source: Regional Guidebook (1998).

Notes: [a] Public School = 0, Roman Catholic or Protestant Christian = 1, Algemeen Bijzonder* = 2.

[b] In the case of comprehensive schools, only the pupils in the grammar stream are considered here.

[c] Comprehensive school = 0; Grammar school = 1.

* These are non-religious private schools.

the school management) for different schools can vary and are important. These inputs include the category of schools (Roman Catholic, Protestant Christian, public school, etc.); whether the school is an exclusive grammar school or a comprehensive school; and the location of schools in the city of The Hague. In the DEA analysis below, we do not consider these non-discretionary inputs for the calculation of performance assessment measures (i.e., efficiency scores). However, they are considered in the study of their influence on the efficiency scores using regression analysis. This is similar to the approach used in the previous application.

The performances of these 46 schools were analyzed using DEA. As mentioned earlier, we considered that the cost per pupil to be approximately the same in the Netherlands, and hence used as imput a dummy variable which has the same value for all the schools.[1] DEA analysis has been carried out using the software package from the University of Warwick (Windows version 1.03). The performance efficiencies of the schools and other relevant

[1] In fact, the simplifying assumption of dummy identical inputs to all schools in the CCR model converts it effectively into the corresponding BCC version (Lovell and Pastor 1999). This makes the model to be insensitive to assumptions on returns to scale. To illustrate this, let us consider the output-oriented envelopment DEA model. The model is,

$$\begin{aligned}
&\max \phi_m \\
&\sum_{n=1}^{46} NO_DELAY_n \lambda_n \geq \phi_m NO_DELAY_m \\
&\sum_{n=1}^{46} EXTRA_SUBJ_n \lambda_n \geq \phi_m EXTRA_SUBJ_m \\
&\sum_{n=1}^{46} AVG_GRADE_n \lambda_n \geq \phi_m AVG_GRADE_m \\
&\sum_{n=1}^{46} INPUT_n \lambda_n \leq INPUT_m \\
&\lambda_n \geq 0;\ \phi_m \text{ free}
\end{aligned}$$

As all inputs are the same, the last constraint becomes $\sum_{n=1}^{46} \lambda_n \leq 1$ which is a restriction for NIRS. Further, Lovell and Pastor (ibid.) have proved that at the optimal solution the strict equality holds, i.e., $\sum_{n=1}^{46} \lambda_n = 1$ making the model coincide with its corresponding BCC version.

results are presented in Table 6.7. The results were insensitive to assumptions on returns to scale.

Table 6.7 Results of the DEA study on school performance evaluation

School	*Efficiency (%)*	*NO-DELAY*		*EXTRA-SUBJ*		*AVG-GRADE*	
		Target	*% to Gain*	*Target*	*% to Gain*	*Target*	*% to Gain*
A1	82.19	63.3	32	46.8	234	6.0	0
A3	83.56	64.3	21	47.6	376	6.1	0
A5	90.41	69.6	10	51.5	222	6.6	0
D6	93.15	71.7	9	53.1	131	6.8	0
D10	89.04	68.6	23	50.8	103	6.5	0
D11	86.30	66.5	41	49.2	54	6.3	0
K14	90.41	69.6	105	51.5	368	6.6	0
L16	100.00	76.0	0	65.0	0	6.9	0
L17	93.15	71.7	59	53.1	11	6.8	0
L18	82.76	65.3	42	52.0	0	5.8	0
L19	91.78	70.7	31	52.3	481	6.7	0
L23	87.67	67.5	27	50.0	–	6.4	0
L27	91.78	70.7	41	52.3	302	6.7	0
LD28	87.67	67.5	14	50.0	16	6.4	0
N31	90.41	69.6	45	51.5	415	6.6	0
N34	86.30	66.5	21	49.2	–	6.3	0
NR36	90.41	69.6	48	51.5	472	6.6	0
O37	89.04	68.6	7	50.8	218	6.5	0
R38	86.30	66.5	28	49.2	97	6.3	0
R40	84.93	65.4	15	48.4	61	6.2	0
H48	89.04	68.6	11	50.8	81	6.5	0
H52	100.00	77.0	0	57.0	0	7.3	0
H53	100.00	79.0	0	63.0	0	7.0	0
H54	91.78	70.7	102	52.3	647	6.7	0
H55	83.56	64.3	114	47.6	90	6.1	0
H57	76.71	59.1	74	43.7	–	5.6	0
H60	86.30	66.5	102	49.2	159	6.3	0
H62	87.67	67.5	65	50.0	–	6.4	0
H66	84.93	65.4	23	48.4	–	6.2	0
H67	89.04	68.6	49	50.8	154	6.5	0
H68	84.93	65.4	33	48.4	86	6.2	0
H69	90.41	69.6	78	51.5	243	6.6	0
H70	90.41	69.6	29	51.5	145	6.6	0
H74	87.67	67.5	78	50.0	525	6.4	0
H75	90.41	69.6	51	51.5	72	6.6	0
S76	88.33	69.0	0	53.3	788	6.3	0

School	Efficiency (%)	NO-DELAY		EXTRA-SUBJ		AVG-GRADE	
		Target	% to Gain	Target	% to Gain	Target	% to Gain
V77	82.19	63.3	54	46.8	569	6.0	0
V79	84.93	65.4	52	48.4	868	6.2	0
V80	82.19	63.3	17	46.8	61	6.0	0
VH82	93.15	71.7	43	53.1	212	6.8	0
W85	90.41	69.6	29	51.5	56	6.6	0
W86	86.30	66.5	7	49.2	17	6.3	0
Z87	79.45	61.2	206	45.3	56	5.8	0
Z91	86.30	66.5	9	49.2	884	6.3	0
Z92	89.66	69.3	24	52.0	0	6.5	0
Z94	90.72	70.0	0	52.2	190	6.6	0

Three schools, L16, H52 and H53 have achieved 100 per cent efficiency. Note that these are the three schools which registered best performances in terms of the variables, EXTRA-SUBJ, AVG-GRADE and NO-DELAY, respectively. Note also that all these are exclusive grammar schools, and the only three grammar schools to be considered in the analysis. These three schools seem to be far more efficient than others, as the next best school (VH82) is only 93 per cent efficient (relative to the performance of these three best schools). The average efficiency achieved is about 88.34 per cent. Customarily, the average efficiency is calculated *excluding* the most efficient performance; this average is about 87.53 per cent. There is no significant variation between the two averages, mainly because the number of the most efficient schools is very small (3 out of 46). Note that the proportion of efficient units reported in many DEA studies is much higher. For example, the numbers of efficient units were 31 out of 55 (Bessent and Bessent 1980); 49 out of 207 (Chalos and Cherian 1995); and 92 out of of 182 banks (Golany and Storbeck 1999) in some of the DEA studies. The relatively low number of efficient units in the present study may be due to the absence of any input variable.

The average efficiency of schools was the highest for the schools located in Leiden (91 per cent), while Voorburg recorded the lowest average efficiency (83 per cent). Though Voorhout registered higher efficiency, it had only one school.

Table 6.7 also gives additional information regarding the source of inefficiencies. It lists the improvements possible in the performance of inefficient schools so that they can be as efficient as their peers. For example, the table specifies that the target for the school A1 as regards the NO-DELAY criterion is 63 per cent. This means that the school A1 can become the most efficient if the percentage of pupils passing without delay is increased from its present level of 48 per cent to 63 per cent. The target (63) represents an increase of 32 per cent over the actual achievement. The target for this school under the EXTRA-SUBJ criterion is 47 per cent, which means that it can become the best-performing school if it can increase the percentage of pupils passing in extra subjects from the present 14 per cent to 47 per cent. Note that there is no slack in terms of the criterion AVG-GRADE for any of the schools, meaning that all the schools performed equally well in terms of average grade.

6.2.2.1 Sensitivity analysis of DEA results: As discussed earlier, a key feature of DEA is that the efficient frontier is formed by the best-performing units (schools here). This is in contrast to techniques such as regression analysis, which seek to average out stochastic error terms. This feature can be a source of a problem in DEA, because there is no direct way of assessing whether a school's deviation from the frontier is statistically significant or not. Hence, it is expected that the robustness of the DEA results be tested using some form of sensitivity analysis.

According to the DEA technique, it is possible for a school to become efficient if it achieves exceptionally better results in terms of one output but performs below average in terms of other outputs. An easy way to test these kind of efficient units is by identifying the peers for inefficient units. If the unit is genuinely efficient, it is expected that there are some inefficient units in its vicinity, so that it is considered a peer for these inefficient units. However, if the unit is not a peer for any inefficient unit, then its best performance is questionable. Other evidence for establishing the superiority of its performance is necessary. Thus, the analysis of peers is an important sensitivity information for the results of DEA analysis. Such peer analysis has been carried out for the present study. The analysis provided peculiar results. Of the three best schools, H53 formed a peer for 44 inefficient schools, confirming it as an efficient school. H52 formed a peer for three inefficient schools, but L16 did not form a peer for any inefficient school.

However, from Table 6.7, we find that L16 has not registered an extraordinarily large performance in terms of an output and unusually low in terms of other outputs to make it an unusual school.

An alternative way of testing the robustness of the DEA results is by conducting the analysis by omitting an input or output and then studying the results. We carried out such analysis, and the results indicated no dramatic changes in the efficiency pattern. For example, when the variable NO-DELAY is removed from the output list, the resulting efficiency pattern showed that H53, which has registered the best performance based on this output, lost its status of 100 per cent efficiency; but the efficiency patterns of other schools did not vary significantly.

Finally, noting that all the three best schools are exclusively grammar schools, we carried out an additional DEA analysis excluding these three. The purpose of this analysis was two-fold. One was to assess the sensitivity of the DEA results for this change; and more importantly to find the best-performing schools among the grammar streams of the comprehensive schools. The results of this analysis show that, of the 43 schools considered, seven schools were ranked the best. The best schools in this case are located in Delft (1), Leiden (2), Voorhout (1), Wassenaar (1) and Zoetermeer (2). No school in The Hague has been considered the best in this case.

6.2.2.2 Regression analysis of the DEA efficiencies: As mentioned earlier, a regression analysis is attempted to find out the effects of non-discretionary inputs on the DEA efficiencies. Specifically, the objectives of the regression exercise were to check whether the impact of the non-discretionary inputs on the efficiency scores were significant, and if so, to understand the nature and extent of their influence on the efficiencies. We considered the following inputs for the purpose of analysis.

(*a*) The category of the schools. We classified the schools as public schools (denoted as PUBLIC), Roman Catholic and/ or Protestant Christian (RKPC) schools and Algemeen Bijzonder (ALGE) schools.

(*b*) Size of the schools (expressed in terms of the number of pupils; refers only to pupils of the grammar stream in the case of comprehensive schools) (denoted as PUPILS).

(*c*) Location of schools within the city of The Hague (denoted as THE HAGUE).
(*d*) Specialization of schools (exclusively grammar schools or grammar streams of comprehensive schools) (denoted as GRAMMAR). We have already established that specialization has a very significant impact on efficiency as all the three best schools are exclusively grammar schools. We have included this variable in the regression analysis for the sake of completeness.

We ran ordinary least squares regression separately for each input, with efficiency score as the dependent variable, using the model,

$$EFF = (a * VARIABLE) + INTERCEPT$$

where *EFF* is the efficiency score and *VARIABLE* refers to inpts such as RKPC, THE HAGUE, etc. as specified above. The results are summarized in Table 6.8.

Table 6.8 Results of the regression analysis with efficiency score as the dependent variable

VARIABLE	*PUPILS*	*GRAMMAR*	*RKPC*	*PUBLIC*	*ALGE*	*THE HAGUE*
INTERCEPT	83.94[a]	87.53[a]	86.70[a]	88.88[a]	88.73	88.09[a]
	(72)[b]	(156)	(84)	(107)	(116)[a]	(101)
Coefficient '*a*'	0.013[a]	12.47[a]	2.90[c]	–1.91[d]	–2.53[d]	0.76[d]
	(4.38)	(5.7)	(2.1)	(1.2)	(1.3)	(0.5)
Multiple *R*-squared	0.304	0.42	0.09	0.03	0.037	0.006
F-statistic	19.22	32.08	4.45	1.49	1.676	0.253
p-value	0.0001	0	0.04	0.23	0.202	0.62

Notes: [a] Significant at 1 per cent confidence level.
[b] Figures within the parentheses indicate the *t*-statistics.
[c] Significant at 5 per cent confidence level.
[d] Insignificant.

The results indicate that efficiency is greatly influenced by the size of the school (measured in terms of the number of pupils in the grammar stream), and that larger schools tend to be more efficient.

As expected, the variable GRAMMAR is also highly significant. Its coefficient can be loosely interpreted to mean that there is an additional 12 per cent efficiency for exclusive grammar schools.

Similarly, Roman Catholic and/or Protestant Christian schools seem to have a significant positive influence on the efficiency score. However, other categories do not seem to have a significant influence, though the direction of this insignificant influence seems to be negative.

Finally, the presence of schools in the city of The Hague does not have any significant influence on their efficiency. However, the direction of influence seems to be positive.

Further comments on DEA using this application are provided in Chapter 7.

6.2.3 Comparative Risk Assessment of Energy Systems

A novel use of DEA to assess selected energy technologies has been presented by Ramanathan (2001b).

Comparative Risk Assessment (CRA) is the balancing of the benefit-cost-risk estimates of all the alternatives for accomplishing the same end purpose (Starr and Whipple 1991). Risk is generally defined as the potential exposure to a loss created by a hazard. A hazard is a situation (physical or societal) which, if encountered, could initiate a range of undesirable consequences. Risk assessment is the process of obtaining quantitative estimate of a risk (probability and consequences). For ease of comparison, CRA techniques generally rely on converting the risks to a single quantitative measure, though opposition does exist to such an approach (Hansson 1989).

A number of studies in the area of CRA are available in the literature (e.g., Rasmussen 1981; Nathwani et al. 1992; see also Ramanathan 2001b for further references). In this section, we discuss the study by Nathwani et al. (1992). The choice is guided by the strong quantitative approach to CRA adopted in the study, and presentation of the comparative numerical data for eight important energy technologies.

According to the authors (ibid.), for discussion of world energy policy, the risks associated with various ways of developing energy supply technologies must be expressed in forms comparable with each other and with the safety benefits indirecty associated with wealth creation in general, and energy use in particular. This has been done by reducing all risks and safety benefits to the common measures of loss or gain of life expectancy (LLE or GLE). The study has considered LLE or GLE which would result if 20 per cent of the total energy supply of a population in the high-income category (with Canada as a typical representative) of the total world population were obtained from the technology considered. In addition, the expected land use and carbon dioxide (CO_2) emissions resulting from the use of technologies have also been estimated. Their estimates of the risk and benefit parameters and other variables associated with different energy supply technologies are given in Table 6.9. For example, according to Nathwani et al. (1992), if the solar photovoltaic option is used to supply 20 per cent of the total energy supply to the high-income category of the world population, it will result in a level of risk equivalent to LLE of one day, require 630 km^2 of land, and release 600 tonnes of CO_2 per GW per year. Note that though solar photovoltaic (PV) energy does not result in net carbon emission while in operation, it does result in carbon dioxide emission in the life-cycle context as the production of the materials needed for solar photovoltaic power plant will result in CO_2 emissions.

Table 6.9 Comparison of impact of different energy supply technologies

Technology Supplying 20% of Total Demand	*Loss of Life Expectancy (days) (LLE)*	*Gain of Life Expectancy (days) (GLE)*	*Land Use (km^2) (LAND)*	*CO_2 Emissions (tonnes) per GW per Year (CO_2)*
Solar photovoltaic	1.0	62	630	600
Biomass	3.5	62	25,600	600
Windmills	1.0	62	9,900	600
Hydroelectric	2.3	62	7,600	2,000
Oil	4.5	62	20	7,00,000
Natural gas	0.8	62	20	4,00,000
Coal	8.4	62	35	9,00,000
Nuclear	0.8	62	10	2,400

Source: Nathwani et al. (1992).

DEA has been used to quantitatively assess the comparative risks of the technologies provided in Table 6.9. Of the four variables used to assess the positive and negative impacts associated with the eight energy supply technologies, LAND is the input, while LLE, GLE and CO_2 are the outputs. Further, as LLE and CO_2 are not the desired outputs, we associate a negative sign with their values. Wherever a range is specified in the original study, we have considered the average values.

6.2.3.1 DEA results and sensitivity analysis: First, a straightforward application of DEA to the data given in Table 6.9 is attempted. The relative efficiency scores are given in Table 6.10.

Table 6.10 Efficiency scores of energy supply technologies

Technology	*Score* (%)
Hydroelectric	1.94
Biomass	2.46
Windmills	6.36
Coal	28.57
Natural gas	50.00
Oil	50.00
Nuclear	100.00
Solar photovoltaic	100.00

It is important to note that the efficiency ratings shown are only *relative*, with regard to the best technology considered. The most efficient technologies is assigned an efficiency score of 100 per cent, while the ratings of others represent their relative ranking, with regard to the best technology.

Note that nuclear and solar photovoltaic energies are rated to be the most efficient technologies. These are followed by natural gas and oil, which are rated as only half as efficient. The main risk associated with nuclear technology is radioactivity. The original study had considered this risk in their calculation of LLE. However, we have found that the relative efficiency for nuclear energy is 100 per cent even if its LLE is increased. This means that in spite of radioactivity problems, nuclear technologies are rated well for the technologies and variables considered in this study. Once the value is changed from 0.8 to 1.0, the only change observed was that the efficiency score of natural gas increased from 50 to 100.

Table 6.10 highlights the major problems associated with renewable technologies. They require a very large land area, thus reducing their efficiency score. On further analysis, it has been found that if the land required for hydroelectric power generation is reduced to about 148 km^2, it becomes 100 per cent relatively efficient. This means that smaller scale hydroelectric plants may be preferable. Hydroelectric technology can become relatively the most efficient if the risks associated it improves so that its LLE increases by about 65 per cent. The results of similar sensitivity analysis pertaining to other technologies for other characteristics are given in Table 6.11. For example, technologies based on natural gas can achieve the best relative efficiency if the land used by them is reduced by 50 per cent or their carbon emissions reduced by more than 99.7 per cent.

Table 6.11 Targets and reduction (percentage) needed for inefficient technologies to reach 100 per cent relative efficiency

Unit	*Land*	*Reduction*	*LLE*	*Reduction*	*CO_2*	*Reduction*
Hydroelectric	147.8	98.06	0.8	65.22	2,000	0.00
Biomass	630.0	97.54	1.0	71.43	600	0.00
Windmills	630.0	93.64	1.0	0.00	600	0.00
Coal	10.0	71.43	0.8	90.48	2,400	99.73
Natural gas	10.0	50.00	0.8	0.00	2,400	99.40
Oil	10.0	50.00	0.8	82.22	2,400	99.66

Further sensitivity analysis is attempted to obtain more insights into comparative risk analysis. The original data pertains to the beginning of the 1990s. The threat of global warming is considered more real and acute at present than at the beginning of this decade. Hence, it will be informative to consider the sensitivity of the efficiency ratings for variations in CO_2 emissions. To assess this, CO_2 emission is excluded from the criterion set; efficiency scores were obtained without this variable. The results showed that, hydroelectric energy, which had higher a CO_2 emission as compared to biomass and windmills, has improved its ranking and moved ahead of them.

We have been witnessing a general reduction in the preference for nuclear technology by many developed countries in the past two decades. Hence, we excluded this option from the set of technologies. The resulting efficiency ratings showed that natural gas and oil technologies are rated the best, along with solar photovoltaic technology. Similarly, when solar photovoltaic technology is excluded, windmills are rated as the best relatively efficient technology, along with nuclear technology.

6.2.4 Energy Efficiencies of Transport Modes in India

The two equally prominent forms of transport—transport of passengers and transport of freight—are measured by composite units, Passenger Kilometres (PKM) and Tonne-Kilometres (TKM) respectively. Both forms of transport require several inputs, such as infrastructure, labour and energy. It is difficult to disaggregate the consumption of these inputs in terms of their use for passenger or freight transport. It can be easily recognized at this stage that DEA provides a natural way of comparing performance efficiencies in such situations.

In this application, one of the most important input as regards transport, namely energy, is considered. The efficiencies of energy utilization by the different transport modes have been assessed using DEA. Normally, the energy consumption figures for transport do not clearly distinguish whether the energy is spent for passenger or freight transport. This is because the same locomotive or vehicle can be used for carrying either passengers or materials, and it may not be possible to disaggregate the total energy consumption in terms of exclusive use for passenger or freight transport. Though it is possible to estimate the energy efficiencies of both forms of transport by apportioning energy consumption on the basis of some norms, the accuracy of these estimates depends upon the norms. Instead, the assessment of 'true' efficiency is possible if a holistic methodology, which considers both passenger and freight performance, is used. DEA provides such a holistic methodology, and has been used here to estimate the relative energy efficiencies of transport in India and trace the pattern of change in these efficiencies over the years.

The trends in energy consumption and physical performance of rail and road transport in India are presented in Table 6.12. Railways have registered negative growth in energy consumption, while registering positive growth in physical performance. The reason is the shift from the fuel-inefficient coal traction to more efficient diesel and electric traction. Similar, straightforward computation of the efficiency of road transport is not possible as sufficient data is not available.

Table 6.12 Physical performance and energy consumption in road and rail transport in India

Year	Railways			Roadways		
	Energy (TJ)	*PKM (billion)*	*TKM (billion)*	*Energy (TJ)*	*PKM (billion)*	*TKM (billion)*
1980–81	2,76,547	209	159	4,42,590	353	98
1981–82	2,88,503	221	174	4,59,360	377	103
1982–83	2,84,951	227	178	4,98,588	408	106
1983–84	2,75,747	223	178	5,30,511	448	116
1984–85	2,59,278	227	182	5,73,339	486	124
1985–86	2,54,974	241	206	6,27,644	850	193
1986–87	2,33,496	257	223	6,83,686	893	210
1987–88	2,23,517	269	231	7,55,410	980	238
1988–89	2,07,529	264	230	8,29,482	905	275
1989–90	1,96,795	281	237	9,16,163	NA	NA
1990–91	1,83,191	296	243	9,39,671	NA	NA
1991–92	1,74,779	315	257	10,06,974	NA	NA
1992–93	1,51,194	300	258	10,76,886	NA	NA
1993–94	1,25,367	296	257	11,45,102	1,500	350
CARG† 1980–94 (%)	–5.49	2.52	3.49	7.03	10.89	9.52

Sources: Various issues of monthly abstracts of statistics and economic surveys published by the Government of India, and various documents published by the Centre for Monitoring Indian Economy, Mumbai.

Note: † Cumulative Annual Rate of Growth.

Table 6.12 shows that the growth of energy consumption by road transport (7 per cent per year during 1980–94) is less than the growth of PKM performance (11 per cent) and TKM (9.5 per cent), indicating roughly an increase in energy efficiency. A similar observation shows an increase in energy efficiency in rail transport as well. However, it is not possible to compare the improvement in the efficiencies of the two modes because of the different levels of changes in energy consumption and performance. Also, such a simplistic analysis cannot provide a numerical assessment of the patterns of efficiency changes over time. However DEA can help to track the progress of energy efficiency of the two major transport modes, which is presented in Ramanathan (2000).

Table 6.13 shows the efficiency scores when both rail and road transport are considered. Results of only the CRS assumption are shown here, because this assumption seems to be more appropriate for road transport. Road transport has registered much larger performance than rail transport in terms of both PKM and TKM. The results show the performance of rail transport in the year 1993–94 to be the relative best in terms of energy efficiency.

Table 6.13 Trend in energy efficiency of Indian transport (rail and road)

	Efficiency		*Slack* (%)	
Year	(%)	*Energy*	*PKM*	*TKM*
		Rail		
1980–81	36.83	63.17	0.00	14.09
1981–82	37.33	62.67	0.00	10.23
1982–83	38.82	61.18	0.00	10.67
1983–84	39.41	60.59	0.00	8.76
1984–85	42.66	57.34	0.00	8.24
1985–86	46.06	53.94	0.00	1.55
1986–87	53.63	46.37	0.00	0.04
1987–88	58.64	41.36	0.00	1.08
1988–89	62.22	37.78	0.38	0.00
1989–90	69.53	30.47	0.00	2.87
1990–91	78.63	21.37	0.00	5.73
1991–92	87.71	12.29	0.00	6.31
1992–93	96.72	3.28	0.00	0.93
1993–94	100.00	0.00	0.00	0.00
		Road		
1980–81	38.86	61.14	0.00	212.65
1981–82	39.99	60.01	0.00	217.67
1982–83	39.87	60.13	0.00	234.06
1983–84	41.15	58.85	0.00	235.26
1984–85	41.30	58.70	0.00	240.16
1985–86	65.99	34.01	0.00	282.28
1986–87	63.65	36.35	0.00	269.10
1987–88	63.21	36.79	0.00	257.39
1988–89	53.16	46.84	0.00	185.64
1993–94	63.39	36.61	0.00	272.00

The energy efficiency of rail transport in the year 1980–81 was only 37 per cent of its efficiency in 1993–94, while the energy efficiency of road in 1993–94 was only 63 per cent as compared to the relative best efficiency. The results of the sensitivity analysis, shown in the same table, show that to be as energy efficient as in 1993–94, in 1980–81 the railways ought to have reduced its consumption by 63 per cent for the same levels of PKM and TKM performance; or it could have increased its TKM performance by 14 per cent. Note that, to achieve the relative best efficiency, road transport in 1993–94 should have consumed about 36 per cent less energy, or it should have almost tripled its freight performance.

6.2.5 Carbon Dioxide Emissions of Countries

The anticipated catastrophic problems due to the greenhouse effect have created a lot of interest in understanding the patterns of energy consumption and carbon dioxide (CO_2) emissions in the world. Reducing the continued emissions of CO_2 and distributing the reduction among the countries (called burden sharing) are the principal foci of national and international negotiations for the past few years. Carbon dioxide emissions are driven by a multitude of factors, the most prominent being energy consumption from fossil fuels, the level of economic activity, and population.

Patterns in the level of energy consumption, economic activity and CO_2 emissions of a country have been analyzed using suitable indicators. Various studies have reported several interesting indicators. For example, Goldemberg (1996) has suggested the use of energy intensity (the ratio of primary energy consumption to GDP) to analyze the energy consumption trends across countries. Grubler and Nakicenovic (1994) has estimated CO_2 emissions per capita, per unit area and per unit GDP for several countries and regions of the world. This information has been used for analyzing the strategies of individual countries in a multi-criteria setting (Ramanathan 1998). Ringuis et al. (1998) have used indicators such as CO_2 emissions per GDP, GDP per capita, and CO_2 emission per capita to identify the effect of burden sharing on different OECD countries using multi-criteira analysis. In general, most of the indicators can be termed as partial indicators, as they indicate CO_2 emissions as a function of only one parameter (such as population in per capita CO_2 emissions, GDP in carbon intensity, or energy consumption in Carbonization Index). A holistic view is possible if one arrives at an index comprizing all the relevant indicators—population, energy consumption, economic activity and CO_2 emissions. The DEA technique has been used to provide such a holistic evaluation of different countries by Ramanathan (2002). The discussion below is based on this paper.

Four indicators are used for the analysis here: CO_2 emissions per capita (denoted as CO_2 hereafter), fossil fuel energy consumption (FOSS), gross domestic product per capita (GDP), and non-fossil fuel energy consumption (NFOSS). FOSS is the sum of primary consumption of coal, oil and gas, while NFOSS is the sum of primary consumption of electricity and primary consumption of non-conventional energy sources. Note that we prefer countries that emit the least CO_2, have the least FOSS consumption, the largest GDP, and the largest NFOSS consumption. Of the four, CO_2 and FOSS are minimization indicators in the sense that countries that register the lowest values in these indicators are preferred over other countries. Hence, these indicators are considered as the input-side parameters of the DEA program, as they have characteristics of the inputs in DEA terminology. Similarly, the indicators NFOSS and GDP are considered as output-side parameters of the DEA program.

The data for the analysis, given in Table 6.14, have been obtained from the ENERDATA database (NRD-Link 3.01). Countries for which consistent data were not available were excluded from our data set. Totally, 64 countries have been considered in the analysis. DEA analysis has been performed using the software package from the University of Warwick (Windows version 1.03). Constant returns to scale has been assumed. This means that a change in GDP and NFOSS results in a proportional change in CO_2 and FOSS.

The resulting efficiency scores of different countries are indicated in Table 6.15.

Table 6.15 shows that only six countries have been considered efficient for the year 1990. They are India, Luxembourg, Norway, Sudan, Switzerland and Tanzania. Nigeria, Mozambique, Myanmar, Sweden and Bangladesh are the next five ranked countries.

Table 6.14 Data on carbon dioxide emissions, energy consumption and gross domestic product for the year 1990

Unit	*Per capita Carbon dioxide Emissions* (*Tons*)	*Fossil Fuel Energy Consumption per capita* (*Tons of Oil Equivalent*)	*Non-fossil Fuel Energy Consumption per capita* (*Tons of Oil Equivalent*)	*Gross Domestic Product per capita* (*Thousands of US$ at 1987 Price and Exchange Rate*)
Albania	1.8895	0.6401	0.1906	0.6403
Algeria	2.5010	0.9839	0.0180	2.6239
Argentina	3.0540	1.1736	0.1705	3.1499
Australia	15.3783	4.7954	0.3035	13.0697
Austria	7.4902	2.6214	0.7018	17.2012
Bangladesh	0.1487	0.0575	0.1529	0.1792
Belgium	11.5296	3.7261	1.1316	15.8955
Bolivia	0.8310	0.3074	0.1303	0.7367
Brazil	1.4886	0.4798	0.4090	1.9515
Bulgaria	9.3078	2.6596	0.5431	3.1762
Canada	15.0580	5.6410	1.9049	15.8964
Chile	2.5359	0.8553	0.2630	1.9227
China	1.9176	0.5644	0.1863	0.2846
Colombia	1.6745	0.5575	0.2435	1.2091
Czech Republic	14.8731	4.1728	0.3227	3.6801
Denmark	10.4967	3.2072	0.3379	20.5414
Egypt	1.5705	0.5826	0.0362	0.8997
Finland	10.8210	3.5657	2.2189	19.5693
France	6.8756	2.3844	1.6277	17.4846
Ghana	0.1743	0.0611	0.2910	0.3894
Greece	7.1030	2.0541	0.0665	5.9877
Hungary	6.9628	2.2505	0.4744	2.4565
India	0.7276	0.2060	0.2163	0.3744
Indonesia	0.8412	0.3009	0.2391	0.5368
Iran	3.2511	1.3077	0.0224	2.7682
Ireland	8.8132	2.8772	0.0176	10.8042
Israel	7.3123	2.4936	–0.0078	9.0967
Italy	7.0852	2.5405	0.1666	14.5952
Japan	8.6288	3.0008	0.5554	22.9283
Luxembourg	32.0342	8.3269	0.9660	22.5011
Malaysia	3.4500	1.2891	0.1376	2.3010
Mexico	3.5278	1.3149	0.1742	1.8700
Morocco	0.8539	0.2836	0.0179	0.9156

Unit	*Per capita Carbon dioxide Emissions (Tons)*	*Fossil Fuel Energy Consumption per capita (Tons of Oil Equivalent)*	*Non-fossil Fuel Energy Consumption per capita (Tons of Oil Equivalent)*	*Gross Domestic Product per capita (Thousands of US$ at 1987 Price and Exchange Rate)*
Mozambique	0.1180	0.0341	0.4827	0.1153
Myanmar	0.0969	0.0357	0.2277	0.2404
Netherlands	10.9916	4.3472	0.1402	16.2826
New Zealand	7.5807	2.7543	1.3571	10.7824
Nigeria	0.3756	0.1450	0.5921	0.3106
Norway	6.9364	2.7199	2.3702	21.9751
Pakistan	0.5949	0.2067	0.1807	0.3503
Papua New Guinea	0.6847	0.2206	0.3861	0.8025
Peru	0.9899	0.3190	0.2348	0.2539
Philippines	0.7190	0.2139	0.2407	0.6165
Poland	9.3088	2.5371	0.0573	1.5589
Romania	7.1289	2.5463	0.1016	1.4524
Singapore	12.4420	4.9383	0.0000	10.1968
Slovak Republic	10.4399	3.2411	0.7403	3.6217
South Africa	9.8531	2.3417	0.3755	2.5582
South Korea	5.4394	1.8030	0.3529	4.1316
Spain	5.6529	1.8332	0.5069	8.6178
Sudan	0.1763	0.0525	0.3648	0.6841
Sweden	6.2086	2.1297	3.4304	20.0141
Switzerland	6.4052	2.3116	1.4200	28.1136
Syria	2.5553	1.0438	0.0403	1.0397
Taiwan	5.9267	1.8801	0.4682	6.1777
Tanzania	0.0825	0.0293	0.4624	0.1525
Thailand	1.7360	0.5276	0.2746	1.2911
Tunisia	1.5222	0.5674	0.1275	1.3089
Turkey	2.4512	0.7389	0.1640	1.7348
United Kingdom	9.9122	3.3592	0.3343	12.8990
United States	19.2082	6.6786	1.0399	19.6549
Venezuela	3.8000	1.9105	0.1905	2.5370
Vietnam	0.3083	0.0848	0.2924	0.6102
Zaire	0.1189	0.0371	0.2799	0.1896

Source: ENERDATA database (NRD-Link 3.01).

Table 6.15 DEA results for the year 1990 for comparing carbon dioxide emissions of different countries

Country	*DEA Scores*	*DEA Rank*	*Peer(s)*	*% Reduction* CO_2	*% Reduction* *FOSS*	*% Increase* *NFOSS*	*% Increase* *GDP*
Albania	15.51	51	Luxembourg, Norway, Tanzania	84.2	84.5	0.0	0.0
Algeria	23.90	40	Switzerland	76.0	94.1	97.6	0.0
Argentina	23.98	39	Switzerland, Sudan	76.7	94.2	0.0	0.0
Australia	19.39	44	Switzerland, Sudan	80.5	91.1	0.0	0.0
Austria	52.37	21	Switzerland	48.0	53.1	7.6	0.0
Bangladesh	70.08	11	India, Sudan, Tanzania	50.0	29.9	0.0	0.0
Belgium	31.60	31	Switzerland, Sudan	68.7	74.7	0.0	0.0
Bolivia	21.73	41	Norway, Switzerland, Tanzania	75.0	78.3	0.0	0.0
Brazil	43.32	25	India, Sudan, Tanzania	60.0	56.7	0.0	0.0
Bulgaria	8.39	58	Norway, Switzerland, Tanzania	91.4	91.6	0.0	0.0
Canada	25.00	36	Switzerland, Sudan	74.8	91.2	0.0	0.0
Chile	17.56	49	Switzerland, Sudan	84.0	87.9	0.0	0.0
China	43.77	24	India	57.9	68.6	0.0	33.3
Colombia	17.88	48	Switzerland, Sudan	82.4	92.1	0.0	0.0
Czech Republic	5.68	62	Switzerland, Sudan	94.6	94.8	0.0	0.0
Denmark	64.43	14	Luxembourg, Switzerland	35.2	35.6	268.6	0.0
Egypt	13.37	54	Switzerland, Sudan	87.5	97.8	0.0	0.0
Finland	58.24	18	Luxembourg, Norway, Tanzania	41.7	41.8	0.0	0.0
France	61.60	16	Switzerland, Sudan	39.1	86.2	0.0	0.0

Country	*DEA Scores*	*DEA Rank*	*Peer(s)*	*% Reduction* CO_2	*% Reduction* *FOSS*	*% Increase* *NFOSS*	*% Increase* *GDP*
Ghana	60.74	17	Switzerland, Sudan, Tanzania	50.0	39.3	0.0	0.0
Greece	19.24	46	Switzerland	80.3	84.2	200.3	0.0
Hungary	8.23	59	Switzerland, Sudan	91.4	92.4	0.0	0.0
India	100.00	1	India	0.0	0.0	0.0	0.0
Indonesia	33.29	30	India, Sudan, Tanzania	62.5	66.7	0.0	0.0
Iran	19.46	43	Switzerland, Sudan	81.8	97.8	0.0	0.0
Ireland	51.91	22	Luxembourg, Switzerland	47.7	48.1	4911.7	0.0
Israel	40.59	26	Luxembourg, Switzerland	58.9	59.4	–7958.3[a]	0.0
Italy	47.14	23	Switzerland, Sudan	53.5	94.1	0.0	0.0
Japan	62.65	15	Switzerland, Sudan	37.2	94.9	0.0	0.0
Luxembourg	100.00	1	Luxembourg	0.0	0.0	0.0	0.0
Malaysia	15.34	53	Switzerland, Sudan	85.7	93.7	0.0	0.0
Mexico	13.23	55	Switzerland, Sudan	85.7	97.7	0.0	0.0
Morocco	24.72	37	Switzerland, Sudan	77.8	92.4	0.0	0.0
Mozambique	90.78	8	Luxembourg, Tanzania	0.0	9.2	0.0	0.0
Myanmar	89.05	9	India, Sudan, Tanzania	0.0	11.0	0.0	0.0
Netherlands	33.78	29	Switzerland, Luxembourg	66.4	86.2	163.4	0.0
New Zealand	57.52	19	Norway, Tanzania	42.1	42.5	0.0	0.0
Nigeria	93.87	7	India, Tanzania	0.0	6.1	0.0	100.0
Norway	100.00	1	Norway	0.0	0.0	0.0	0.0
Pakistan	26.17	35	India, Sudan, Tanzania	66.7	73.8	0.0	0.0
Papua New Guinea	53.08	20	Luxembourg, Norway, Tanzania	42.9	46.9	0.0	0.0

(*Table 6.15 contd.*)

(*Table 6.15 contd.*)

Country	*DEA Scores*	*DEA Rank*	*Peer(s)*	*% Reduction* CO_2	*FOSS*	*% Increase* *NFOSS*	*GDP*
Peru	7.51	60	Switzerland, Sudan, Tanzania	90.0	92.5	0.0	0.0
Philippines	26.63	34	India, Sudan, Tanzania	71.4	73.4	0.0	0.0
Poland	3.87	64	Switzerland, Sudan	95.7	98.9	0.0	0.0
Romania	4.71	63	Switzerland, Sudan	95.8	98.3	0.0	0.0
Singapore	36.46	27	Luxembourg, Switzerland	63.7	63.5	–[b]	0.0
Slovak	11.71	56	Luxembourg, Norway, Tanzania	88.5	88.3	0.0	0.0
South Africa	6.28	61	Switzerland, Sudan	93.9	96.7	0.0	0.0
South Korea	18.05	47	Switzerland, Sudan	81.5	95.2	0.0	0.0
Spain	35.66	28	Switzerland, Sudan	64.9	90.9	0.0	0.0
Sudan	100.00	1	Sudan	0.0	0.0	0.0	0.0
Sweden	74.54	10	Switzerland, Sudan	25.8	26.5	0.0	0.0
Switzerland	100.00	1	Switzerland	0.0	0.0	0.0	0.0
Syria	9.28	57	Switzerland, Sudan	92.3	95.3	0.0	0.0
Taiwan	24.13	38	Switzerland, Sudan	76.3	89.0	0.0	0.0
Tanzania	100.00	1	Tanzania	0.0	0.0	0.0	0.0
Thailand	19.39	44	Switzerland, Sudan	82.4	92.3	0.0	0.0
Tunisia	19.77	42	Switzerland, Sudan	80.0	83.1	0.0	0.0
Turkey	17.13	50	Switzerland, Sudan	84.0	95.5	0.0	0.0
UK	30.12	32	Switzerland, Sudan	69.7	95.5	0.0	0.0
USA	27.18	33	India, Sudan	72.9	97.5	0.0	0.0

Country	*DEA Scores*	*DEA Rank*	*Peer(s)*	*% Reduction* CO_2	*FOSS*	*% Increase* *NFOSS*	*GDP*
Venezuela	15.46	52	Switzerland, Sudan	84.2	95.4	0.0	0.0
Vietnam	66.85	13	India, Sudan, Tanzania	33.3	33.1	0.0	0.0
Zaire	69.14	12	India, Sudan, Tanzania	0.0	30.9	0.0	0.0

Notes: [a] This value is negative as the primary consumption of electricity for this country is negative.
[b] This value requires division by zero, as the original value of NFOSS is zero.

Central European countries such as Poland, Romania, Czech Republic and Hungary, along with South Africa and Peru, are considered the least efficient according to the DEA analysis.

Table 6.15 also gives information about peer(s) for countries considered inefficient. For example, Austria's peer is Switzerland, meaning that the former can try to emulate the latter in order to register the values of indicators that will be considered best in the DEA study. Note that, in general, peers belong to the same group (OECD/developing country, etc.) as the countries for which they are peers. While Switzerland is chosen as a peer for most of the developed countries, Sudan and Tanzania are considered as peers for most of the developing countries. India is considered as a peer for most developing countries of Asia. Further columns in the table provide the reduction/increase in the values of specific indicators that would make inefficient countries to be deemed efficient by the DEA. For example, if Austria is to be considered efficient, it should register about 48 per cent reduction in its per capita CO_2 emissions (keeping the values of all other indicators constant). Alternatively, it will be considered efficient if it registers about 53 per cent reduction in fossil fuel energy consumption, or about 7.6 per cent increase in its non-fossil fuel energy consumption.

7

Some Additional Discussion on Data Envelopment Analysis

The past chapters have presented the basic and advanced features of DEA. The emphasis in those chapters has been on the computational and software aspects of the technique. While these aspects are important in any DEA application, other behavioural and usage aspects are also equally important. They will be discussed in this concluding chapter of the book. Then, a brief discussion of the strengths and limitations will be provided.

7.1 Some Considerations on the Application Procedure of DEA

While the routine computations of DEA can be performed using general LP software or specialized DEA software, there are several non-computational aspects that are important in the application procedure of DEA. These aspects relate to the choice of DMUs for a given DEA application, selection of inputs and outputs, choice of a particular DEA model (e.g., CRS, VRS, etc.) for a given application, and choice of an appropriate sensitivity analysis procedure. These are briefly discussed in this section. A good discussion of the application procedure for DEA is available in Golany and Roll (1989). The discussion in this section is based on this article and other more recent articles (e.g., Dyson et al. 2001).

7.1.1 Selection of DMUs to be Compared

Two factors influence the selection of DMUs for a study. They are—homogeneity, and the number of DMUs.

(*a*) The DMUs must be homogenous units. They should perform the same tasks, and should have similar objectives. The inputs and outputs characterizing the performance of DMUs should be identical, except for differences in intensity or magnitude. For example, DEA efficiencies will not be appropriate when the performance of universities and secondary schools are compared because their inputs and outputs would be very different.

(*b*) The number of DMUs to be compared depends upon the objectives of the DEA study, and on the number of homogeneous units whose performance in practice has to be compared. However, some considerations have been specified in this selection of the number of DMUs for a DEA study.

(i) If the number of DMUs is high, then the probability of capturing high performance units that determine the efficiency frontier will also be high. A large number of DMUs will also enable a sharper identification of typical relations between inputs and outputs. In general, as the number of DMUs increases, more inputs and outputs can be incorporated in a DEA analysis. However, the DEA analyst should be cautious not to increase the number of units unnecessarily. The most important consideration in the selection of the number of DMUs should be the homogeneity of the DMUs. One should not relax this and include heterogeneous units which are not comparable with the rest just for the sake of increasing the number of DMUs.

(ii) The relation between the number of DMUs and the number of input and output is sometimes specified some rules of thumb.

- The number of DMUs is expected to be larger than the product of number of inputs and outputs (Darrat et al. 2002; Avkiran 2001) in order to discriminate effectively between efficient and inefficient DMUs. However, there are many examples in the literature where DEA has been used with small sample sizes.

- ♦ The sample size should be at least 2 or 3 times larger than the sum of the number of inputs and outputs.

7.1.2 Selection of Inputs and Outputs

A main difficulty in any application of DEA is in the selection of inputs and outputs. The criteria of selection of these inputs and outputs are quite subjective. There is no specific rule in determining the procedure for selection of inputs and outputs. However, some guidelines may be suggested, and are discussed below.

A DEA study should start with an exhaustive, initial list of inputs and outputs that are considered relevant for the study. At this stage, all the inputs and outputs that have a bearing on the performance of the DMUs to be analyzed should be listed. Screening procedures, which may be quantitative (e.g., statistical) or qualitative (simply judgemental, using expert advice or using methods such as the Analytic Hierarchy Process [Saaty 1980]), may be used to pick up the most important inputs and outputs and, therefore reducing the total number to a reasonable level. For the purpose of this filtering, questions such as the following may help:

(*a*) Is the input or output related to one or more of the objectives of the DEA study?

(*b*) Does the input or output identify the characteristics of the DMUs that are not captured by other inputs or outputs?

Normally, inputs are defined as resources utilized by the DMUs or conditions affecting the performance of DMUs, while outputs are the benefits generated as a result of the operation of the DMUs. However, sometimes it may become difficult to classify a particular factor as input or output, especially when the factor can be interpreted either as input or as output. In such cases, one way of classifying the factor to check whether DMUs recording higher performance in terms of that factor is considered more efficient or not. If yes, the factor is normally classified as an output. Otherwise, it is classified as an input.

In any study, it is important to focus on specifying inputs and outputs correctly. For a meaningful study, it is important to restrict the total number of inputs and outputs to reasonable levels. Some rules of thumb specified above can help to determine the appropriate number of inputs and outputs. Usually, as the number of inputs and outputs increases, there will be more number of DMUs that will get an efficiency rating of 1, as they become too specialized to be evaluated with respect to other units. In other words, as mentioned earlier, it is possible for DMUs to concentrate on a few inputs and/or outputs and score highest efficiency ratings, leading to large number of DMUs with unit efficiency ratings. In any study, it is important to focus on correctly specifying inputs and outputs.

7.1.3 Choice of the DEA Model

A variety of DEA models have been presented in the preceding chapters: input maximizing or output minimizing, multiplier or envelopment, constant or variable returns to scale, etc. As we have seen in Chapter 2, the outputs of many of these models are related. However, some considerations may be useful in choosing an appropriate DEA model.

In applications that involve inflexible inputs (not fully under control), output-based formulation would be more appropriate. However, in applications where outputs are decided by the goals of the management rather than by extracting the best possible performance of the DMUs, input-based DEA formulation may be more appropriate. Multiplier versions are used when inputs and outputs are emphasized in an application, while envelopment versions are used when the relations among the DMUs are emphasized.

The choice of constant or variable returns to scale depends on the specific application. When the performances of DMUs are not normally expected to depend on the scale of operation (e.g., comparisons of performance of several large monopolies), constant returns to scale (CRS) seem appropriate. In most of other cases, variable returns to scale (VRS) may be a more appropriate assumption.

7.1.4 Post-DEA Procedures

A DEA study normally provides information about efficiencies of DMUs, slacks, and peers, among others. It is important to verify the robustness of these results using sensitivity analysis. In some cases, these DEA outputs are sufficient for making relevant conclusions; other cases may need further analysis of DEA output. The latter studies will be discussed in this section.

7.1.4.1 Sensitivity analysis of DEA results: DEA is an extreme point technique because the efficiency frontier is formed by the actual performance of best-performing DMUs. A direct consequence of this aspect is that errors in measurement can affect DEA results significantly. DEA efficiencies are very sensitive to even small errors. Furthermore, since DEA is a non-parametric technique, statistical hypothesis tests are difficult. For example, it may not be possible to estimate the confidence with which DEA efficiencies are computed (in the sense of confidence as used in the field of statistics). Hence, as with any modelling technique, the outputs generated by DEA should be viewed with caution, and should be used only after conducting appropriate sensitivity analysis. Some such procedures for DEA have been described in the literature (e.g., Smith and Mayston 1987). Some of the sensitivity analysis procedures have been detailed in Chapter 6, following the discussions of DEA applications.

It is possible for a DMU to obtain a value of utility by simply improving its performance in terms of only one particular output ignoring others. The DMU will be considered efficient even though it has not improved its performance in terms of all the outputs. However, such an unusual DMU will not be a peer for many inefficient units. Thus, if a DMU is initially identified as efficient by DEA, a supplementary sensitivity analysis should be conducted by checking the number of inefficient DMUs for which it is a peer. If the number is high, then the DMU is genuinely efficient; the efficiency of a DMU with only a few peers should always be viewed with caution.

Another way of checking the sensitivity of DEA efficiency of a DMU is to verify whether the efficiency score of a DMU is affected appreciably if only one input or output is omitted from the DEA analysis. An efficient DMU that is ranked inefficient due to the omission of just one input or one output should be viewed with caution. A similar sensitivity analysis should be conducted by leaving out an efficient DMU from the analysis.

7.1.4.2 Using DEA output for further analysis: There are situations where further analysis of DEA efficiencies will be needed. Several DEA studies reported in the literature have used additional methodologies such regression analysis, principal factor analysis, and the Malmquist productivity index approach to analyze DEA output further. Some of the examples reported in Chapter 6 have used regression analysis of DEA efficiencies to filter the effect of uncontrollable factors. Principal Factor Analysis has been employed in conjunction with DEA scores by Yeh (1996). The Malmquist productivity index approach (see Chapter 4) allows DEA efficiencies to be used for time series analysis.

7.2 Strengths and Limitations

Data envelopment analysis is a powerful technique for performance measurement. This is illustrated by the large and growing number of its applications in various fields, some of which have been reported in this book. These applications provide evidences of the strengths of DEA, some of which are discussed below.

(*a*) The main strength of DEA is its objectivity, i.e., DEA provides efficiency ratings based on numerical data, and not by using subjective opinions of people. DEA is certainly a very valuable evaluation tool that makes the maximum possible objective use of the available data. The results of DEA are useful if one accepts the principle of frontier analysis.

(*b*) DEA can handle multiple input and multiple outputs, and they can be measured in very different units. For example, in the example presented in Chapter 1, one input (capital employed) was measured in money units, while the other unit (number of employees) was measured in number units.

(*c*) Unlike statistical methods of performance analysis, DEA is non-parametric in the sense that it does not require an assumption of a functional form relating inputs to outputs.

However, DEA does have certain limitations, which have been discussed in the literature. These are:

(*a*) Application of DEA requires solving a separate linear program for each DMU. Hence, the application of DEA to problems that have many DMUs can be computationally intensive. However, this is not a very serious problem, considering the computational power of present-day computers, and the number of DMUs that are considered in normal DEA problems.

(*b*) Since DEA is an extreme point technique, errors in measurement can cause significant problems. DEA efficiencies are very sensitive to even small errors, making sensitivity analysis an important component of post-DEA procedure. This aspect has already been discussed earlier in this chapter.

(*c*) Since DEA is a non-parametric technique, statistical hypothesis tests are difficult. This issue has also been discussed earlier.

(*d*) As efficiency scores in DEA are obtained after running a number of LP problems, it is not easy to explain intuitively the process of DEA for the case of more than two inputs and outputs to a non-technical audience. A general audience, which will normally not have a background in linear programming, may not consider DEA transparent. In general, the management of the organizations, for which a DEA study will be carried out, may find it difficult to comprehend its results. They sometimes prefer simpler applications, if possible. However, it is possible to explain the process of DEA (and linear programming) in simpler terms, which could help win their support.

Some further important limitations of DEA are summarized below, especially from the point of view of the performance evaluation of schools, presented in Chapter 6 (Section 6.2.2).

(*e*) DEA has been designed to compute efficiency scores only when one or more inputs and one or more outputs are used for the analysis. It would be better if the methodology has the flexibility to allow for one or more or even nil outputs or inputs for performance evaluation. Note that in the performance evaluation of schools, it was felt that the inputs are almost uniform across schools in the Netherlands. Hence, it was assumed that there was no need to use any input measure. However, it was not possible to directly proceed with a DEA analysis, without any input. Though this problem was overcome by introducing one dummy input variable that has the same value for all the schools, it indicated a possible limitation of the methodology.

(*f*) A second problem with DEA is the way in which efficiencies are calculated. The values of weights of inputs and outputs are chosen (by the methodology) as the optimal value of a linear program for each DMU. They are not considered physically significant in the DEA literature. Thus a DEA analysis of schools provided 46 different sets of these weights, but they were not used further. Though some restrictions on weights may be introduced in the form of additional constraints (see Section 4.4.3), the management does not have a direct control on their values even if it wishes to associate measures of importance to minimizing any one or more of inputs or maximizing any one or more of outputs. Clearly, it would be better if the choice of weights lies with the management. This will increase the flexibility of the methodology, and will make the methodology more intuitive for the management.

(*g*) Sometimes, the DEA analysis may lead to unexpected results. This can be explained using a simplified graphical (frontier) analysis of the data of the schools.[1] Figure 7.1 shows the performance 46 schools when only two outputs (NO-DELAY and EXTRA-SUBJ) are used for performance evaluation. Obviously, the schools with higher efficiency are represented by points as far away from origin as possible. The schools L16 and H53 are recognized as the most efficient units by the DEA analysis, and the so-called efficiency frontier is formed by these two schools

[1] See Chapter 1 for a detailed description of graphical (frontier) analysis.

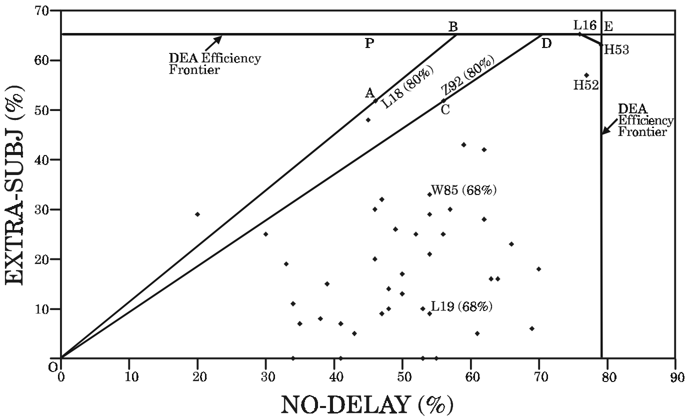


Figure 7.1 A graphical representation of DEA analysis of some schools for the two-output case. The outputs considered are NO-DELAY and EXTRA-SUBJ. Percentage figures within brackets are the DEA efficiencies

and the x- and y-axes. The schools lying within this frontier are less efficient. Take the case of two schools L18 and Z92. Both have the same performance in terms of the criterion EXTRA-SUBJ, while Z92 has done better in terms of the criterion NO-DELAY. In such a case, one would consider Z92 to be more efficient than L18. However, the DEA efficiency scores are the same for both, as the ratio of achieved performance and the best performance is the same for both (i.e., the ratios OA/OB and OC/OD are equal). In fact, DEA analysis will rank any school which attains 65 per cent in EXTRA-SUBJ criterion to be 100 per cent efficient, though its performance in terms of NO-DELAY is low (say just 45 per cent at the point P). Similarly, compare the schools L19 and W85. Both have the same performance in terms of NO-DELAY, though W85 is far superior in terms of EXTRA-SUBJ (33 per cent compared to L19's only 9 per cent). However, DEA analysis provides equal efficiency for both. Obviously, such a DEA analysis will not be acceptable to schools such as Z92 and W85.

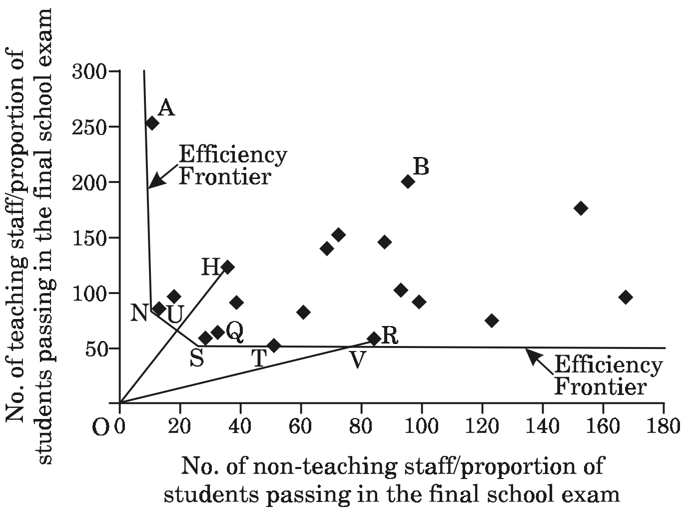
(*h*) As mentioned above, it is possible for any school to attain best efficiency by single-handedly improving its performance in one score and not attaining much in terms of other outputs (for example, the point P in Figure 7.1). This aspect may send wrong signals to schools (or DMUs), which, in their interests to record 100 per cent efficiency, may concentrate on improving their performance in terms of only a few outputs and may not be interested in their overall improvement. Of course, as discussed earlier in this chapter, it is possible to identify such schools/DMUs using post-DEA sensitivity analysis. For example, schools/DMUs which concentrate on improving their performance in terms of only a few outputs can be identified by finding the number of inefficient DMUs for which the DMU is a peer.

Appendix: Solutions to Selected Problems

Exercise 1.5

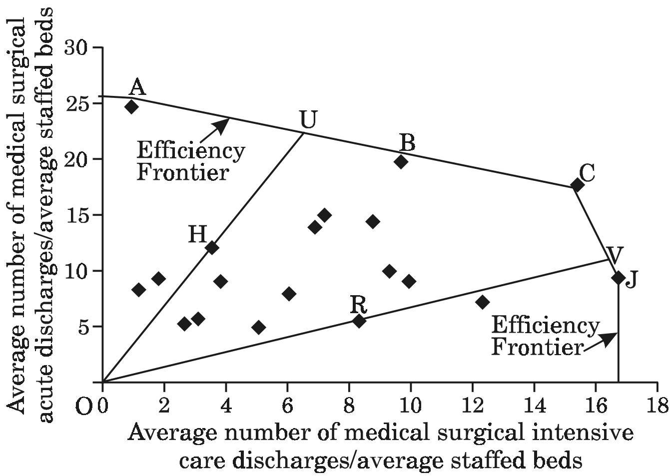
Problem 3

The figure is given below. Schools A, N, S and T are the most efficient schools. Efficiency of School H is given by OU/OH. Similarly, efficiencies for other schools can be estimated.



Problem 4

The figure below shows that A, C and J are the most efficient hospitals. Efficiency of Hospital R is given by OR/OV. Similarly, efficiencies of other hospitals can be estimated.



Exercise 2.2

Problem 5

The following are the efficiencies of the schools.

School	*Efficiency* (%)
A	100.00
B	27.79
C	29.21
D	37.43
F	37.13
G	39.65
H	56.35
I	50.68
J	53.12
K	82.96
L	54.83
M	65.65
N	100.00
O	65.36
P	67.26
Q	94.99
R	89.64
S	100.00
T	100.00

Problem 6

The following are the efficiencies of the hospitals.

Hospital	*Efficiency (%)*
A	100.00
B	98.89
C	100.00
D	73.29
F	74.34
G	68.69
H	54.48
I	60.25
J	100.00
K	39.89
L	62.81
M	43.03
N	34.80
O	43.65
P	74.04
Q	28.55
R	50.52
S	26.99
T	32.28

Exercise 2.7

Problem 7

The following is the input minimizing multiplier program for Firm B.

$$\begin{aligned}
&\min \quad 2.5u'_{Inp,B} \\
&\text{subject to} \\
&12v'_{Out,B} = 1 \\
&5v'_{Out,B} - u'_{Inp,B} \le 0 \\
&12v'_{Out,B} - 2.5u'_{Inp,B} \le 0 \\
&11v'_{Out,B} - 2u'_{Inp,B} \le 0 \\
&40v'_{Out,B} - 6u'_{Inp,B} \le 0 \\
&80v'_{Out,B} - 10u'_{Inp,B} \le 0 \\
&v'_{Out,B}, u'_{Inp,B} \ge 0
\end{aligned}$$

Efficiency for Firm B is 60 per cent. Firm E is its peer.

Exercise 3.6

Problem 4

(*a*) Yes
(*b*) Yes
(*c*) No
(*d*) Yes
(*e*) No

Problem 5

(*a*) No. They are efficient only under CRS. M, Q and R are also efficient under VRS.
(*b*) Yes
(*c*) No. Project to C. An input-oriented CRS DEA program for Firm A will project it on to the point C.
(*d*) No
(*e*) Yes
(*f*) Yes, it will be NDRS
(*g*) For Firm A,

CRS efficiency = FC/FA
VRS efficiency = FB/FA, and
Scale efficiency = FC/FB,

where F is the projection of A on the *Y*-axis.

Problem 6

The following table provides the efficiency values of the power generation companies under assumptions of Constant (CRS) and Variable Returns to Scale (VRS).

UNIT	*CRS*	*VRS*
CHUBU	95.88	98.20
CHUGOKU	97.04	97.75
DENGEN-KAIHATSU	100.00	100.00
FUKUI	87.53	100.00
FUKUYAMA	97.63	100.00
HOKKAIDO	96.77	97.02

UNIT	*CRS*	*VRS*
HOKURIKU	92.10	92.17
JYOUBAN	94.40	95.12
KANSAI	92.31	95.39
KASHIMA	93.16	94.21
KIMITSU	94.61	96.02
KYUSHU	100.00	100.00
MIZUSHIMA	97.19	100.00
OKINAWA	100.00	100.00
OOITA	100.00	100.00
SAKAI	82.81	100.00
SAKATA	100.00	100.00
SHIKOKU	94.40	94.50
SUMITOMO	83.01	87.63
TOBATA	95.06	95.87
TOHOKU	100.00	100.00
TOKYA	97.39	100.00
TOMAKOMAI	90.64	100.00
TOYAMA	94.79	98.15
WAKAYAMA	95.52	100.00

Problem 7

The following table provides the efficiency values of the airlines under the assumptions of Constant (CRS) and Variable Returns to Scale (VRS).

Airline	*CRS*	*VRS*
1	100.00	100.00
2	84.39	100.00
3	73.73	83.71
4	72.64	79.62
5	82.56	87.36
6	94.75	100.00
7	100.00	100.00
8	97.66	98.10
9	84.64	100.00
10	80.40	81.53
11	87.33	90.76
12	100.00	100.00
13	90.91	91.49
14	100.00	100.00
15	84.45	85.86

Problem 8

The following table provides the efficiency values of the companies under the assumptions of Constant (CRS) and Variable Returns to Scale (VRS).

Company Name	*CRS*	*VRS*
Ahawmut Natl. Corp.	85.24	93.63
AMR Corp.	100.00	100.00
Arvin Industries Inc.	49.96	51.46
Bank of Boston Corp.	73.51	96.17
Bankers Trust New York	51.43	51.77
Barnett Banks Inc.	42.66	47.25
Chase Manhattan Corp.	52.27	64.31
Chrysler Corp.	100.00	100.00
Comerica Inc.	50.61	79.53
Continental Bank Corp.	44.30	81.16
Corestates Financial Corp.	63.20	64.15
Dana Corp.	57.59	75.44
Delta Air Lines Inc.	49.61	57.02
E-Systems Inc.	100.00	100.00
Eaton Corp.	54.15	65.24
First of America Bank Corp.	55.45	56.66
First Union Corp.	100.00	100.00
Ford Motor Co.	67.36	73.07
Gencorp Inc.	38.01	44.23
General Dynamics Corp.	27.55	48.38
General Motors Corp.	45.17	48.29
KeyCorp	90.01	100.00
Lockheed Corp.	45.93	100.00
Martin Marietta Corp.	66.46	66.76
McDonnell Douglas Corp.	23.04	26.24
Mellon Bank Corp.	47.84	48.58
Meridian Bancorp Inc.	70.08	75.34
Paccar Inc.	37.27	39.04
Raytheon Co.	84.51	86.01
Republic New York Corp.	100.00	100.00
Sequa	74.51	76.58
Southwest Airlines Co.	92.62	100.00
Suntrust Banks	38.03	45.04
The Boeing Co.	42.64	42.95
UAL	76.85	98.73
US Air Group Inc.	58.42	100.00

Problem 9

CRS and VRS efficiencies of the banks are given in the table below.

Bank	*CRS*	*VRS*
1	84.7	84.76
2	89.91	100.00
3	94.29	95.51
4	100.00	100.00
5	100.00	100.00
6	100.00	100.00
7	100.00	100.00
8	97.84	100.00
9	100.00	100.00

Problem 10

The following table provides the efficiency values of the restaurant branches under the assumptions of Constant (CRS) and Variable Returns to Scale (VRS).

Branch	*CRS*	*VRS*
1	83.05	90.12
2	86.41	100.00
3	51.49	56.18
4	100.00	100.00
5	45.37	46.97
6	100.00	100.00
7	99.05	100.00

References

Adolphson, D.L., Cornia, G.C. and Walters, L.C. (1989), Railroad property valuation using data envelopment analysis, *Interfaces*, 19 (3), 18–26.

Agrell, Per J. and Tind, Jørgen (1998), *Extensions of DEA-MCDM Liaison*, Working Paper 3/1998, Department of Operations Research, University of Copenhagen, Denmark.

Ali, Agha Iqbal (1994), Computational aspects of DEA, In: Charnes, A., Cooper, W.W., Lewin, A.Y. and Seiford, L.M. (Eds.), *Data Envelopment Analysis: Theory, Methodology, and Application*, Kluwer Academic Publishers, Massachusetts, USA, 63–88.

Ali, Agha Iqbal and Seiford, L.M. (1993), Computational accuracy and infinitesimals in Data Envelopment Analysis, *INFOR*, 31 (4), 290–97.

Anderson, D.R., Sweeney, D.J. and Williams, T.A. (2000), *An Introduction to Management Science: Quantitative Approaches to Decision Making*, 9th edition, South Western Publishing, Cincinatti, USA.

Avkiran, Necmi K. (2001), Investigating technical and scale efficiencies of Australian Universities through data envelopment analysis, *Socio-Economic Planning Sciences*, 35, 57–80.

Banker, R.D. (1984), Estimating most productive scale size using data envelopment analysis. *European Journal of Operational Research*, 17, 35–44.

Banker, R.D., Charnes, A. and Cooper, W.W. (1984), Some models for estimating technical and scale efficiencies in Data Envelopment Analysis, *Management Science*, 30 (9), 1078–92.

Banker, R.D. and Morey, R.C. (1986), Efficiency analysis for exogenously fixed inputs and outputs, *Operations Research*, 34 (4), 513–21.

Bates, J.M., Baines, D.B. and Whynes, D.K. (1996), Measuring the efficiency of prescribing by general practitioners, *Journal of the Operational Research Society*, 47, 1443–51.

Belton, V. and Stewart, T.J. (1999), DEA and MCDA: Competing or Complementary Approaches? In: Meskens, N. and Roubens, M. (Eds.), *Advances in Decision Analysis*, Kluwer Academic Publishers, 87–104.

Bessent, Authella M. and Bessent, Wailand E. (1980), Determining the comparative efficiency of schools through data envelopment analysis, *Educational Administration Quarterly*, 16 (2), 57–75.

Bessent, A.M., Bessent, W., Elam, J. and Long, D. (1984), Educational Productivity council employs management science methods to improve educational quality, *Interfaces*, 14 (6), 1–8.

Bessent, A.M., Bessent, W., Kennington, J. and Reagan, B. (1982), An application of mathematical programming to assess productivity in the Houston Independent School District, *Management Science*, 28, 1355–67.

Brockett, P.L., Charnes, A., Cooper, W.W., Huang, Z.M. and Sun, D.B. (1997), Data transformations in DEA cone ratio envelopment approaches for monitoring bank performances, *European Journal of Operational Research*, 98, 250–68.

Caves, D.W., Christensen, L.R. and Diewert, W.E. (1982), Multilateral comparisons of output, input and productivity using index numbers, *Economic Journal*, 92, 73–86.

Chalos, P. and Cherian, J. (1995), An application of data envelopment analysis to public sector performance measurement and accountability, *Journal of Accounting and Public Policy*, 14, 143–60.

Charnes, A., Cooper, W.W. and Rhodes, E. (1978), Measuring the efficiency of decision making units, *European Journal of Operations Research*, 2, 429–44.

——— (1979), Short communication: measuring the efficiency of decision making units, *European Journal of Operational Research*, 3, 339–339.

Charnes, A., Clark, Charles T., Cooper, W.W. and Boaz, Golany (1985a), A developmental study of data envelopment analysis in measuring the efficiency of maintenance units in the U.S. Air Forces, In: Russell, G. and Robert Thrall, G. (Eds.), *Annals of Operations Research*, 2 (1), 95–11.

Charnes, A., Cooper, W.W., Golany, B., Seiford, L. and Stutz, J. (1985b), Foundations of data envelopment analysis for Pareto-Koopmans efficient empirical production functions, *Journal of Econometrics*, 30 (1/2), 91–107.

Charnes, A., Cooper, W.W., Huang, Z.M. and Sun, D.B. (1990), Polyhedral cone ratio DEA models with an illustrative application to large commercial banks, *Journal of Econometrics*, 46, 73–91.

Charnes, A., Cooper, W.W., Seiford, L. and Stutz, L. (1982), A multiplicative model for efficiency analysis, *Socio-Economic Planning Sciences*, 16, 223–24.

——— (1983), Invariant multiplicative efficiency and piecewise Cobb-Douglas envelopments, *Operations Research Letters*, 2 (3), 101–3.

Charnes, A., Cooper, W.W., Wei, Q.L. and Huang, Z.M. (1989), Cone ratio data envelopment analysis and multi-objective programming, *International Journal of Systems Science*, 20, 1099–1118.

Charnes, Abraham, Cooper, William W. and Lewin, Arie Y. (1994), *Data envelopment analysis: Theory, methodology and applications,* Kluwer, Boston.

Coelli, T. (1996), *A Guide to DEAP Version 2.1: A Data Envelopment Analysis (Computer) Program*, CEPA Working Paper 96/08, Centre for Efficiency and Productivity Analysis, University of New England, Australia.

Cooper, William W., Seiford, Lawrence M. and Tone, Kaoru (2000), *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*, Kluwer Academic Publishers, Boston, USA.

CIRT (1995), Performance Statistics of State Transport Undertakings—1993–94, Central Institute of Road Transport, Pune, India.

Darrat, Ali F., Can Topuz and Tarik Yousef (2002), *Assessing Cost and Technical Efficiency of Banks in Kuwait*, Paper presented to the ERF's 8th Annual Conference in Cairo, ERF, Cairo, Egypt (*http://www.erf.org.eg/html/Finance_8th/Assessingcost-Darrat&Yousef.pdf*).

Dyer, James S., Fishburn, Peter C., Steuer, Ralph E., Wallenius, Jyrki and Zionts, Stanley (1992), Multiple criteria decision making, multi-attribute utility theory: the next ten years, *Management Science*, 38 (5), 645–54.

Dyson, R.G., Allen, R., Camanho, A.S., Podinovski, V.V., Sarrico, C.S. and Shale, E.A. (2001), Pitfalls and protocols in DEA, *European Journal of Operational Research*, 132, 245–59.

Dyson, R.G. and Thanassoulis, E. (1988), Reducing weight flexibility in data envelopment analysis, *Journal of the Operational Research Society*, 39 (6), 563–76.

Emrouznejad, A. (1995–2001), Ali Emrouznejad's DEA HomePage, Warwick Business School, Coventry, UK (*www.deazone.com*).

Fare, R. and Grosskopf, S. (1985), A nonparametric approach to scale efficiency, *Scandinavian Journal of Economics*, 87, 594–604.

Fare, R, Grosskopf, S. and Lee, W.F. (1995), Productivity in Taiwanese manufacturing-industries, *Applied Economics,* 27, 259–65.

Fare, R., Grosskopf, S. and Lovell, C.A.K. (1985), *The Measurement of Efficiency of Production*, Kluwer Nijhoff, Boston, USA.

——— (1994), *Production Frontiers*, Cambridge University Press.

Fare, R., Grosskopf, S. and Tyteca, D. (1996), An activity analysis model of the environmental performance of firms—application to fossil-fuel-fired electric utilities, *Ecological Economics*, 18, 161–75.

Farrel, M.J. (1957), The measurement of productive efficiency, *Journal of Royal Statistical Society (A)*, 120, 253–81.

Ganley, J.A. and Cubbin, J.S. (1992), *Public Sector Efficiency Measurement: Applications of Data Envelopment Analysis*, North-Holland, Amsterdam.

Golany, B. (1988), An interactive MOLP procedure for the extension of DEA to effectiveness analysis, *Journal of the Operational Research Society*, 39, 725–34.

Golany, B. and Roll, Y. (1989), An application procedure for DEA, *Omega,* 17 (3), 237–50

Golany, B. and Storbeck, J.E. (1999), A data envelopment analysis of the operational efficiency of bank branches, *Interfaces*, 29 (3), 14–26.

Golany, Boaz and Thore, Sten (1997), The economic and social performance of nations: efficiency and returns to scale, *Socio-Economic Planning Sciences*, 31 (3), 191–204.

Goldemberg, J. (1996), A note on the energy intensity of developing countries, *Energy Policy*, 24 (8), 759–61.

Good, D. and Rhodes, E. (1990), *Productive efficiency, technological change, and the competitiveness of U.S. airlines in the Pacific Rim*, Paper presented at the Transportation Research Forum, Long Beach, California.

Grubler, Arnulf and Nakicenovic, Nebojsa (1994), *International Burden Sharing in Greenhouse Gas Reduction*, RP-94-9, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Halme, M., Joro, T., Korhonen, P., Salo, S. and Wallenius, J. (1999), A value efficiency approach to incorporating preference information in Data Envelopment Analysis, *Management Science*, 45, 103–15.

Hansson, S.O. (1989), Dimensions of risk, *Risk Analysis*, 9 (1), 107–13.

Herrero, Inés and Pascoe, Sean (2002), Estimation of technical efficiency: a review of some of the stochastic frontier and DEA software, *Computer in Higher Education and Economics Review* (CHEER), 15 (1), University of Bristol, UK (*http://www.economics.ltsn.ac.uk/cheer/ch15_1/dea.htm*).

Hjalmarsson, Lennart and Odeck, James (1996), Efficiency of trucks in road construction and maintenance: An evaluation with data envelopment analysis, *Computers and Operation Research*, 23 (4), 393–404.

Jablonský, J. (1999). A spreadsheet based support for data envelopment analysis, In: Plesinger, J. (Ed.), *Proceedings of the 17th International Conference on Mathematical Methods in Economics*. Jindrichuv Hradec, Republic of Czechoslovakia, 133–38.

Joro, T., Korhonen, P. and Wallenius, J. (1998), Structural comparison of data envelopment analysis and multiple objective linear programming, *Management Science*, 40, 962–70.

Lootsma, Freerk A. and Ramanathan, R. (1999), Some experiences on assessment of schools in the Netherlands: a comparison between DEA and MCDA, Paper presented at the Seventh IIASA/DAS Workshop on 'The Future of Decision Analysis and Support', International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria, 6–7 September 1999.

Lovell, Knox, C.A. and Pastor, Jesús T. (1999), Radial DEA models without inputs or without outputs, *European Journal of Operational Research*, 118, 46–51.

Majumdar, Sumit K. (1997), Modularity and productivity: Assessing the impact of digital technology in the U.S. telecommunications industry, *Technological Forecasting and Social Change*, 56, 61–75.

Malmquist, S. (1953), Index numbers and indifference surfaces, *Trabajos de Estatistica*, 4, 209–42.

Maxwell, Daniel T. (2000), Decision analysis: aiding insight V—fifth biennial survey, *ORMS Today*, 27 (5), 29–35 (*http://www.lionhrtpub.com/software-surveys.shtml*).

Nakayama, Hirotaka, Tetsuzo Tanino and Yeboon Yun (2000), Generalized data envelopment analysis and its applications, In: Shi, Yong and Milan, Zeleny (Eds.), *New Frontiers of Decision Making for the Information Technology Era*, World Scientific Publishing Co. Pte. Ltd., Singapore, 227–48.

Nathwani, J.S., Siddall, E. and Lind, N.C. (1992), *Energy for 300 Years: Benefits and Risks*, Institute for Risk Research, University of Waterloo, Waterloo, Ontario, Canada.

Norman, M. and Stoker, B. (1991), *Data Envelopment Analysis: The Assessment of Performance*, John Wiley & Sons, Chichester, UK.

Nozick, L.K., Borderas, H. and Meyburg, A.H. (1998), Evaluation of travel demand measures and programs: A Data Envelopment Analysis approach, *Transportation Research A*, 32 (5), 331–43.

Olesen, O.B. and Petersen, N.C. (1995), Incorporating quality into data envelopment analysis: A stochasic dominance approach, *International Journal of Production Economics*, 39, 117–35.

——— (1996), A presentation of GAMS for DEA, *Computers and Operations Research*, 23 (4), 323–39.

Oum, T.H. and Yu, C. (1991), *Economic efficiency of passenger railway systems and implications on public policy*, Working Paper 91-TRA-013, Faculty of Commerce and Business Administration, The University of British Columbia, Vancouver, Canada.

Premachandra, I., Powell, J.G. and Shi, J. (1998), Measuring the relative efficiency of fund management strategies in New Zealand using a spreadsheet-based stochastic data envelopment analysis model, *Omega*, 26 (2), 319–31.

Productivity Commission (1999), *An Assessment of the Performance of Australian Railways, 1990 to 1998*, Supplement to Inquiry Report Progress in Rail Reform, AusInfo, Canberra, November.

Proudlove, Nathan (2000), *Using Excel for Data Envelopment Analysis*, Working Paper No. 2007, Manchester School of Management, Manchester, UK.

Ramanathan, R. (1998), A Multi-Criteria Methodology to the Global Negotiations on Climate Change, *IEEE Transactions on Systems, Man and Cybernetics—Part C: Applications and Reviews*, 28 (4), November, 541–48.

——— (1999), Using Data Envelopment Analysis for assessing the productivity of the State Transport Undertakings, *Indian Journal of Transport Management*, 23 (5), 301–12.

——— (2000), A holistic approach to compare energy efficiencies of different transport modes, *Energy Policy*, 28 (11), 743–47.

Ramanathan, R. (2001a), A Data Envelopment Analysis of Comparative Performance of Schools in the Netherlands, *OPSEARCH—The Indian Journal of Operational Research*, 38 (2), 160–82.

——— (2001b), Comparative Risk Assessment of Energy Supply Technologies: A Data Envelopment Analysis Approach, *Energy—The International Journal*, 26, 197–203.

——— (2002), Combining Indicators of energy consumption and CO_2 emissions: A cross-country comparison, *The International Journal of Global Energy Issues*, 17 (3), 214–22.

Rasmussen (1981), The application of probabilistic risk assessment techniques to energy technologies, *Annual Review of Energy*, 6, 123–38.

Ray, Subash C. (1991), Resource-use efficiency in public schools: A study of Connecticut data, *Management Science*, 37 (12), 1620–28.

Regional Guidebook (1998), *Regional Guidebook with Quality Cards of Secondary Schools. Region 8—The Hague, Leiden, Delft, Zoetermeer* (in Dutch) Sdu Service Centre, The Hague, The Netherlands.

Ringuis, L., Torvanger, A. and Holtsmark, B. (1998), Can multi-criteria rules fairly distribute climate burdens? OECD results from three burden sharing rules, *Energy Policy*, 26 (10), 777–93.

Saaty, Thomas L. (1980), *The Analytic Hierarchy Process*, McGraw-Hill, New York.

Samuelson, Paul A. and Nordhaus, Willian D. (1989), *Economics*, Thirteenth Edition, McGraw-Hill, New York.

Seiford, Lawrence M. (1994), A DEA bibliography (1978–1992), In: Charnes et al. (Eds.), 437–69.

——— (1996), Data envelopment analysis: The evolution of the state of the art (1978–1995), *The Journal of Productivity Analysis*, 7, 99–137.

Seiford, Lawrence M. and Thrall, Robert M. (1990), Recent developments in DEA: The mathematical programming approach to frontier analysis, *Journal of Econometrics*, 46, 7–38.

Seiford, Lawrence M. and Zhu, Joe (1999), An investigation of returns to scale in data envelopment analysis, *Omega, The International Journal of Management Science*, 27, 1–11.

Shafer, Scott M. and Byrd, Terry A. (2000), A framework for measuring the efficiency of organizational investments in information technology using data envelopment analysis, *Omega*, 28, 125–41.

Smith, Peter and Mayston, David (1987), Measuring efficiency in the public sector, *Omega—International Journal of Management Science*, 15 (3), 181–89.

Starr, Chauncey and Whipple, Chris (1991), The strategic defence initiative and nuclear proliferation from a risk analysis perspective, In: Shubik, Martin (Ed.), *Risk, Organizations, and Society*, Kluwer Academic Publishers, Boston, 49–61.

Stewart, T.J. (1992), A critical survey on the status of multiple criteria decision making theory and practice, *Omega*, 20 (5/6), 569–86.

Stewart, T.J. (1996), Relationships between data envelopment analysis and multicriteria decision analysis, *Journal of the Operational Research Society*, 47 (5), 654–65.

Sueyoshi, Toshiyuki and Goto, Mika (2001), Slack-adjusted DEA for time series analysis: Performance measurement of Japanese electric power generation industry in 1984–1993, *European Journal of Operational Research*, 133, 232–59.

Taha, H.A. (1997), *Operations Research—An Introduction*, Prentice-Hall Inc., New Jersey, USA.

Thanassoulis, Emmanuel (1999), Data Envelopment Analysis and Its Use in Banking, *Interfaces*, 29 (3), 1–13.

Thanassoulis, E. and Dyson, R.G. (1992), Estimating preferred target input-output levels using data envelopment analysis, *European Journal of Operational Research*, 56, 80–97.

Thompson, R.G., Singleton Jr., F.D., Thrall, R.M. and Smith, B.A. (1986), Comparative site evaluations for locating a high-energy physics lab in Texas, *Interfaces*, 16, 35–49.

Tulkens, H. (1993), On FDH efficiency: some methodological issues and applications to retain banking, courts and urban transit, *Journal of Productivity Analysis*, 4, 183–210.

Walden, John B. and Kirkley, James E. (2000), *Measuring Technical Efficiency and Capacity in Fisheries by Data Envelopment Analysis Using the General Algebraic Modeling System (GAMS): A Workbook*, NOAA Technical Memorandum NMFS-NE-160, National Oceanic and Atmospheric Administration, US Department of Commerce, Massachusetts, USA (*http://www.nefsc.nmfs.gov/nefsc/publications/tm/tm160/tm160.htm*).

Yeh, Q. (1996), The application of data envelopment analysis in conjunction with financial ratios for bank performance evaluation, *Journal of the Operational Research Society*, 47, 980–88.

Zhu, J. (1996), Data envelopment analysis with preference structure, *Journal of the Operational Research Society*, 47, 136–50.

Zhu, Joe (2002), *Quantitative Models for Performance Evaluation and Benchmarking: DEA with spreadsheets and Excel Solver,* Kluwer Academic Publishers, Boston.

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